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Aspects that Affect Whole Number Learning: Cultural Artefacts and Mathematical Tasks

Veronica Sarungi
The Aga Khan University, veronica.sarungi@aku.edu

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Chapter 9
Aspects that Affect Whole Number Learning: Cultural Artefacts and Mathematical Tasks

Maria G. Bartolini Bussi, Maitree Inprasitha, Ferdinando Arzarello, Hyman Bass, Ulrich Kortenkamp, Silke Ladel, Caroline Lajoie, Yujing Ni, Thomas Rottmann, Veronica Sarungi, Sophie Soury-Lavergne, and Jenny Young-Loveridge

9.1 Introduction

9.1.1 What Was Presented at the Conference: Overview

In this chapter, discussion of key socio-cultural aspects that affect learning will be considered from two complementary perspectives:

- Aspects that may help learning, especially if adequately exploited by the teacher.
- Aspects that may hinder learning, especially if not adequately contrasted by the teacher.

Thirteen papers written by authors from ten countries were accepted for Theme 3. For presentation and discussion, the papers accepted for Theme 3 were divided into subgroups according to their main focus. We are aware that it is not possible to make distinct groups of papers as there are several overlaps in the classification, but in

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The first two authors wrote the chapter with the collaboration of other working group members.

M.G. Bartolini Bussi (✉)
Department of Education and Humanities, University of Modena and Reggio Emilia, Modena, Italy
e-mail: mariagiuseppina.bartolini@unimore.it

M. Inprasitha (✉)
Khon Kaen University, Khon Kaen, Thailand
e-mail: inprasitha_crme@kku.ac.th

F. Arzarello
University of Turin, Turin, Italy

H. Bass
University of Michigan, Ann Arbor, MI, USA

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order to focus the group discussion, the main ideas from the papers are used to dictate the different time slots.

9.1.1.1 Language and Institutional Contexts

The contribution of participants from many different contexts offers a unique possibility to have first-hand reports about issues which may foster or hinder the construction of mathematical meanings.

Transparency vs Opacity Some papers addressed the transparency of language for Chinese (Ni 2015), Thai (Inprasitha 2015) and Maori (Young-Loveridge and Bicknell 2015) that was contrasted with the opacity of European languages such as French and German (Peter-Koop et al. 2015).

Pimm and Sinclair (2015) analysed the grammar of 20 different languages about fractions, discussing the information conveyed by each of them.

The Institutional Context Mercier and Quilio (2015) analysed the differences between primary school education about whole number arithmetic in four French-speaking countries, showing that language is only one of the variables to be considered when addressing the functioning principles of education systems.

9.1.1.2 Artefacts

A cluster of papers addressed different kinds of artefacts:

- The number line (Bartolini Bussi 2015; Electronic Supplementary Material: Bartolini Bussi 2017).
- Tallies and sequences of tallies (Hodgson and Lajoie 2015).

U. Kortenkamp
Universität Potsdam, Potsdam, Germany

S. Ladel
Universität des Saarlandes, Saarbrücken, Germany

C. Lajoie
Université du Québec à Montréal, Montreal, QC, Canada

Y. Ni
Chinese University of Hong Kong, Hong Kong, China

T. Rottmann
Bielefeld University, Bielefeld, Germany

V. Sarungi
Institute for Educational Development, The Aga Khan University, Dar es Salaam, Tanzania

S. Soury-Lavergne
IFE ENS de Lyon, Lyon, France

J. Young-Loveridge
University of Waikato, Hamilton, New Zealand

- Multibase arithmetic blocks and arithmetic rack or Slavonic abacus (Rottmann and Peter-Koop 2015; Electronic Supplementary Material: Rottmann 2017).
- Cuisenaire rods (Ball and Bass 2015), with a special challenging task for economically disadvantaged fifth graders.
- Artefacts from everyday life (Inprasitha 2015).
- Computer games (Bakker et al. 2015).
- Virtual manipulatives (Soury-Lavergne and Maschietto 2015; Ladel and Kortenkamp 2015).

9.1.1.3 Teacher Education

An overarching issue that encompasses the conditions for the learning of WNA concerns teachers' education and the effects it may have on their future students' processes. All the above papers, in some sense, hint at the importance of teacher education for the effective use of either language or other artefacts.

Two specific programmes for teacher education were reported:

- An approach developed in Canada (Laval University, Québec) for establishing the foundations of WNA with pre-service primary school teachers that highlights the role of mathematicians in the preparation of teachers in arithmetic and that also stresses the complementary role of mathematicians and mathematics educators in such an endeavour (Hodgson and Lajoie 2015)
- A programme developed in Thailand in order to adapt the Japanese Lesson Study to the Thai context (Inprasitha 2015, Electronic Supplementary Material: Inprasitha 2017)

9.1.2 The Discussion in the Working Group

The eight one-hour sessions were organised in different ways. At the beginning, two small groups were organised: (1) language to focus on different wording of 'ten' and (2) artefacts and mathematics to focus on relationships between epistemology and the choice/design of artefacts.

The language group discussed language for grouping, unitising into units and groups. In some countries (e.g. in England), the word 'unit' is used to refer to ones, as well as being a general collective term for different groups (e.g. hundreds, tens, ones). In some languages (e.g. French, German, Italian), there is a particular term used to label the unit for a particular group (e.g. *dizaine*, *zehner* or *decina* for ten). In contrast, English uses ten for the number of items and for the unit name. Moreover, they also discussed about language and wording in fractions and cardinal and ordinal names.

The artefact group discussed both traditional and ICT artefacts. They expressed the need to clarify the terminology according these questions: What is a representation? What is a model? What is an artefact? What is a tool? Furthermore, they discussed about artefacts which are designed and used with different intentions.

They stressed the importance of guidance (by the teacher) and the exploration of the artefacts by the students.

In the last session, there was only a mathematical task group.

The CANP observer, Veronica Sarungi (Tanzania), offered a lively report on the problems in South-East Africa, where the local languages are in conflict, in most cases, with the school language (see also this volume, Chap. 3). The three young observers from the Great Mekong Area (Weerasuk Kanauan, Visa Kim and Chanhpheng Phommaphasouk) were very active in videotaping all the sessions and preparing the report of the group discussion.

At the end the participants agreed that the *language* issue could have been enriched by the confrontation with participants in other working groups, in order to exploit the presence of more linguistic contexts (see Chap. 3). The participants rather chose to focus on *artefacts* (that was considered the true core of the working group discussion) and on *mathematical tasks*, whose careful choice may well foster or hinder learning of WNA. For the artefacts, they expressed the hope to collect together examples of artefacts presented in other working groups. This collective choice, reported in plenary session, determined the structure of this chapter.

9.1.3 The Structure of This Chapter

The core of this chapter is the notion of artefact, from the discussion of the meaning of the word in the literature to a gallery of cultural artefacts from the participants' reports and the literature. The use of cultural artefacts as teaching aids is then addressed. A special section is devoted to the artefacts (teaching aids) from technologies.

The issue of tasks was simply skimmed, as it was not possible to discuss about artefacts without considering the way of using artefacts with suitable tasks. There was no intention to overlap with the ICMI Study 22 (Watson and Ohtani 2015) that was attended by some participants in the ICMI Study 23, including the co-chairs (only the volume of proceedings was available at the time of the Conference). Some examples of tasks were reported to elaborate on aspects that may foster learning WNA, and some examples of tasks that might hinder learning were also reported. Artefacts and tasks appear as an inseparable pair, to be considered within the system of cultural and institutional constraints.

In the concluding remarks, some challenges are outlined, in order to contend with this complex map.

9.2 Cultural Artefacts

9.2.1 The Use of Different Terms with Similar (Not the Same) Meaning

9.2.1.1 The Historic-Cultural School

In the literature, many different words have been used to describe artefacts. One of the difficulties comes from the existence of literature in different languages, with various challenges of translation. The case of translations from Vygotsky's original papers in Russian is emblematic. Vygotsky is the founder of the so-called historic-cultural school, where the notion of mediation by cultural artefacts is central. According to Russian scholars (Anna Stetsenko, personal communication), the major term used by Vygotsky in his papers is *sign* (or *symbol* interchangeably), in Russian знак (*znak* in transliteration), so that the construct of semiotic mediation is expressed in this way '*znakovaya kulturnaya mediatsija*' (знаковая, культурная медиация), meaning symbolic cultural mediation. In the English translations, several different terms were used with related yet different meanings.

In 1930, Vygotsky gave a talk on The Instrumental Method in Psychology at the Krupskaya Academy of Communist Education that was later included in different readings. The English version of the transcript reads:

> In the behavior of man we encounter quite a number of artificial devices for mastering his own mental processes. By analogy with technical devices these devices can justifiably and conventionally be called psychological tools or instruments. […] Psychological tools are artificial formations. By their nature they are social and not organic or individual devices. They are directed toward the mastery of [mental] processes – one's own or someone else's – just as technical devices are directed toward the mastery of processes of nature. The following may serve as examples of psychological tools and their complex systems: language, different forms of numeration and counting, mnemotechnic techniques, algebraic symbolism, works of art, writing, schemes, diagrams, maps, blueprints, all sorts of conventional signs, etc. (Rieber and Wollock 1997, p. 85)

In this translation, different kinds of terms are used: (psychological) tool, instrument, artificial formation or device. The idea of 'artificial device', whence the short name 'artefact', was used by Yrio Engeström (1987) and subsequently by Michael Cole (1996). Cole argued in favour of using 'artefacts', as a more generic term (1996, p. 108). Cole connected artefact mediation to Dewey's analysis of tools and works of art, claiming that Dewey's works were 'well known among Russian educators and psychologists' (p. 109). This direct scientific connection between Dewey and Vygotsky is intentional, as Cole claims that his 'focus will be on an attempt to formulate an approach to psychology that draws upon both national traditions' (p. 115). This choice makes Vygotskian ideas closer to US scholars, but not everybody agrees on the mutual consistency of the two national traditions. For instance, Stetsenko (2008) writes:

> Whereas both Dewey and Piaget (and many of their contemporary followers in the relational ontology approach) treated human beings as no different than other biological

> organisms—thus keeping up with the Darwinian notion that 'nature makes no drastic leaps'— Vygotsky and his followers postulated precisely such a leap and turned to exploring its implications. In doing so, these scholars followed with the Marxist dialectical materialist view according to which "…[the] base for human thinking is precisely *man changing nature* and not nature alone as such, and the mind developed according to how human being[s] learned to change nature" (Engels quoted in Vygotsky 1997, p. 56; italics in the original). (p. 482)

In the same manner, Xie and Carspecken (2007), in their comparative analysis of US and Chinese mathematics curricula, contrast Dewey and Marx as representatives of two very different paths of departure from Hegelian idealism with strong influence on educational choices.

The notion of artefact was elaborated in the so-called activity theory approach (e.g. Engeström 1987) and exploited in mathematics education by other authors (see Bartolini Bussi and Mariotti 2008 for a review).

In the further literature on the instrumental approach, an artefact is meant 'as the – often but not necessarily physical – object that is used as a tool' (Hoyles and Lagrange 2010, p. 108), while:

> [an instrument requires a relationship] between the artefact and the user for a specific type of task. Besides the artefact, the instrument also involves the techniques and mental schemes that the user develops and applies while using the artefact. To put it in the form of a somewhat simplified 'formula' we can state: Instrument = Artefact + Schemes and techniques, for a given type of task. (p. 108)

In this case, the reference is to Rabardel's (1995) instrumental approach.

In this chapter, however, we shall not strictly use this distinction, and in most cases we shall refer to artefacts in a more generic way, as mathematics educators, anthropologists and historians do not always adhere to either of the above theoretical frameworks.

9.2.1.2 The Theory of Semiotic Mediation: The Teacher's Side

The notion of artefact (in the Vygotskian sense) is central in the *theory of semiotic mediation* as developed by Bartolini Bussi and Mariotti (2008). In this framework, there are two main foci: the function of cultural artefacts, developed by mankind, and the teacher's role as cultural mediator.

The teacher is in charge of two main processes: the *design of activities* and the *functioning of activities*. In the former, the teacher makes sound choices about the artefacts to be used, the tasks to be proposed and the pieces of mathematics knowledge to be addressed, taking into account the curricular choices. This means that mathematics knowledge is, in this framework, the *taught knowledge* to be distinguished from the *scholarly knowledge* (Chevallard and Bosch 2014). In the latter, the teacher exploits, monitors and manages the children's observable processes, to decide how to interact with them and what and how to fix in the individual and group memory. The design process is encapsulated by the left triangle of Fig. 9.1, where the *semiotic potential* of the artefact is described. The semiotic potential concerns the double semiotic link defined by the use of the artefact to accomplish

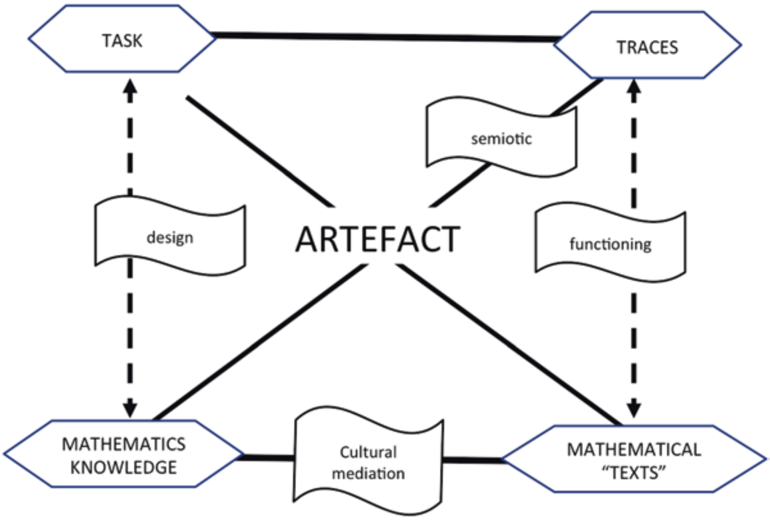


Fig. 9.1 Semiotic mediation

the task and the mathematical meanings related to the artefact and its use. The other parts of the scheme concern the *functioning in the classroom*. When students are given a task, they start a rich and complex semiotic activity, producing traces (gestures, drawings, oral descriptions and so on). The teacher's job is first to collect all these traces (observing and listening to children), to analyse them and to organise a path for their development towards mathematical 'texts' that can be put in relationship with the pieces of mathematics knowledge into play. In this process, the teacher organises the alternation (in what is called a *didactical cycle*) of individual and small group use of the artefact and production of signs to solve the task and of large group mathematical discussion.

In this way, artefacts become *teaching aids* as they are used in knowledge transmission (e.g. in schools) with didactical intentions. In the literature very often some kinds of teaching aids are named *manipulatives*, in order to highlight the possibility to manipulate them in the process of construction of mathematical meanings (Bartolini Bussi and Martignone 2014; Nührenbörger and Steinbring 2008). Recently, many *virtual manipulatives* have been produced thanks to the increasing distribution of technologies, sometimes without a careful investigation of the cognitive difference (and distance) between a direct manipulation and a mediated manipulation, for instance, by means of the digital mouse (http://nlvm.usu.edu/en/nav/vlibrary.html). For relevant exceptions, see Sect. 9.3.4.

9.2.1.3 Artefacts and Representations: The Learner's Side

When artefacts are in play, what is in the foreground is not the mathematical concept itself but an external *representation* (or a *model*) of it. In this sense, artefacts 'allow children to establish connections between their everyday experiences and their nascent knowledge of mathematical concepts and symbols' (Uttal et al. 1997, p. 38). Nevertheless, the role of an artefact to be a representation is not necessarily obvious for children. In this context, Uttal et al. (1997, p. 43) mention a 'dual-representation hypothesis': Any artefact can be thought of as a representation that stands for something else or as an object on its own. The latter view might provide a reason for children's learning difficulties:

> Concrete objects can help children gain access to concepts and processes that might otherwise remain inaccessible. However, there is another side to the use of concrete objects: children may easily fail to appreciate that the manipulative is intended to represent something else – that it is a symbol. If so, the manipulative will be counterproductive. (Uttal et al. 1997, p. 52)

Similar concerns about the focus of attention in using manipulatives are expressed by Nührenbörger and Steinbring (2008), with the same problem applying to virtual manipulatives as well (see Sect. 9.3.4).

Monaghan, Trouche and Borwein (2016) have an ambitious approach with a coverage from prehistory to future directions in the field, with a major emphasis on modern technologies, addressing the areas of curriculum, assessment and policy design.

After this short review about the function of artefacts in the literature on mathematics education, it is worthwhile to present some examples of cultural artefacts, drawing on the examples mentioned by the participants.

9.2.2 Cultural Artefacts for WNA

The history of mathematics is replete with the creation of artefacts, some disseminated all over the world, while others are related to a particular culture. Hence, cultural artefacts are important in both the history and geography of mathematics and reveal something about the cultures that have produced and used them, as well as about the image of mathematics in that culture. Some of them may be exploited to reconstruct the cultural identity of learners or to construct mathematical concepts.

According to Vygotsky's list quoted in Sect. 9.2.1, language is the first example of artefact (sign) directed towards the mastery of mental processes. Language is at work in both everyday and school contexts. The connection between language and numbers (including whole number arithmetic) is far from being natural or universal. In this volume (Chaps. 3 and 5), the variation of different forms of numeration and counting are explored with reference to the history and geography of whole number arithmetic. In some cases language can foster learning; in some cases it can hinder learning. In this chapter, we analyse a typical example where language (and culture) makes the difference (see the case of *epistemological obstacle* in Sect. 9.3.2). Language enters also in the activity with other artefacts, when tasks are given by language or to be answered by language. The gallery of examples is organised in the following way:

- Ancient artefacts to represent numbers and compute (tallies, counting rods, *quipus* and *yupana*)
- Abaci

- Artefacts for multiplication (pithy tables, Napier bones, ‘gelosia’ scheme)
- The number line
- Songs, poems and dance
- Games
- Everyday artefacts
- Textbooks and d-books

9.2.2.1 Ancient Cultural Artefacts to Represent Numbers and Compute

Tallies (this volume, Sects. 5.2.3.1 and 10.4.1) are, according to historians (Menninger 1969), the most ancient representations of numbers (Fig. 9.2).

Tallies are still used in election poll counting (Fig. 9.3).

Tallies were used for centuries also in double tally sticks (Menninger 1969, p. 223) for commercial exchanges:

> A long piece of wood is cut lengthwise almost the end; the part with the large end is the ‘stock’ (the main stick) and the split-off portion is the ‘inset’ (the piece laid on the main stock). […] When a payment or delivery is either made or received, the debtor insert his inset in the stock, which the creditor generally keeps, and notches are cut into or removed from both pieces at once. Then both parties take back their own pieces and keep them until the final settlement. In this marvelously simple fashion, the ‘double bookkeeping’ makes any cheating impossible. (p. 231)

According to Menninger (1969, p. 233), the Chinese character for ‘contract’ (契, qìjù) is very meaningful.

> The word for ‘contract’ in Chinese is symbolized by two characters at the top, one for a tally stick (stick with notches) and one for a knife, and another at the bottom which means ‘large’. A ‘contract’ or ‘agreement’ in Chinese is thus literally a ‘large tally stick’.

Sequence of tallies is among the founding elements of an arithmetic course that was developed in Canada (Laval University, Québec) for the preparation of pre-service elementary school teachers (this volume, Chap. 10). In that course, tallies are used to fully define natural numbers and operations on those numbers; capture

Fig. 9.2 The Ishango bone (http://www.cs.mcgill.ca/~rwest/link-suggestion/wpcd_2008-09_augmented/images/234/23448.jpg.htm)

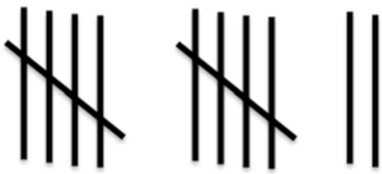


Fig. 9.3 Tallies for ballot counts

Fig. 9.4 Chinese counting rods from excavation

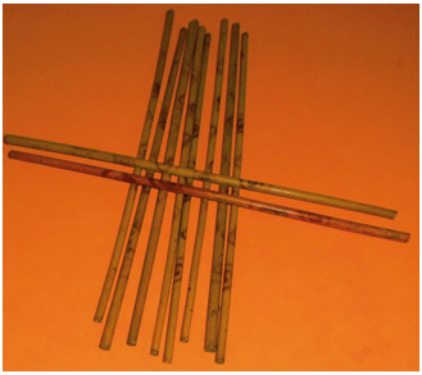


Fig. 9.5 Chinese counting rods from Mekong area (personal collection of the first author)

the notion of equality; prove some fundamental properties, such as the commutativity of addition; and more (Hodgson and Lajoie 2015).

Counting rods (筹; *chóu*, this volume, Chaps. 3 and/or 5) were used in ancient China (Zou 2015), with the possibility of distinguishing positive (red) and negative (black) numbers too, and gave rise also to the ancient Chinese characters for numbers (Fig. 9.4) (see this volume, Chap. 3).

Later they were spread all over the world and are one of the most effective strategies to introduce place value by means of bundles. Figure 9.6 shows an ancient method textbook for teachers, published in Italy in 1920. A comparison between Figs. 9.5 and 9.6 shows that the Chinese rods are bamboo, while the Italian sticks represent other European species of trees.

Quipu (González and Caraballo 2015) is a system of strings (Fig. 9.7), used by Incas, with different colours and different knots, where the position of knots and the colours of the strings determine the number to be represented. According to Jacobsen (1983):

> Documented evidence, however, provides that early Hawaiians and ancient Chinese predated the Incan usage. Studies concentrating on the quipu as an accounting device rather than as an element in the evolution of the writing process might provide valuable contributions to the solution of the mystery surrounding this artifact. Insight into the development of mankind in the Pacific may be gained by understanding the use of the quipu in the East and West, and in Hawaii—the "meeting place" of the Pacific. (p. 53)

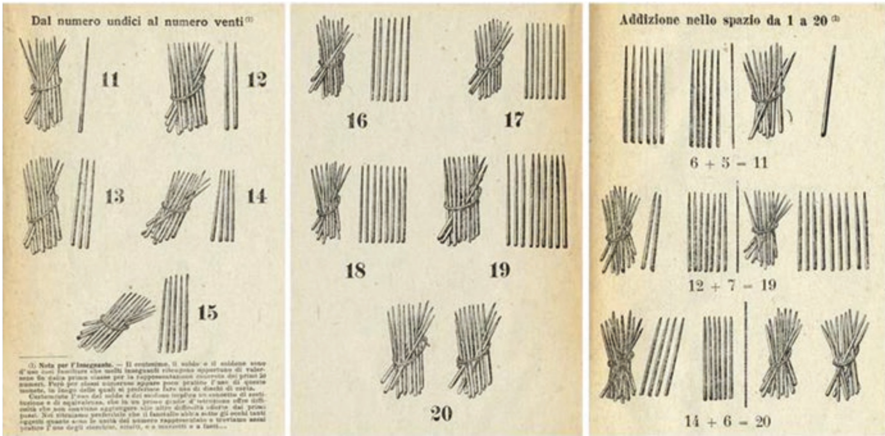


Fig. 9.6 Counting rods from Conti (1920)

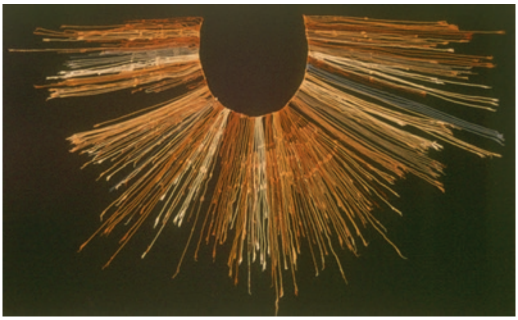


Fig. 9.7 An Inca *quipu* from the Larco Museum in Lima (https://commons.wikimedia.org/wiki/File:Inca_Quipu.jpg)

In Fig. 9.8, besides *quipus* also a *yupana* is represented. According to Gonzales and Caraballo (2015), it draws on a base-ten system. It was used by Incas to do arithmetical operations. It is still used as a teaching aid in Peru in intercultural education programmes.

9.2.2.2 Abaci

Different kinds of *abaci* are present in the history and geography of arithmetic. The Roman *abacus*, *suàn pán* (算盘), *soroban* (そろばん) and *schoty* (счёты) share some features of bead arithmetic:

- In each column, one bead is equal to ten beads of the adjacent column on the right.
- Each column is divided into two parts: each bead of the top part is equivalent to five beads on the bottom.

Fig. 9.8 *Quipu* and *yupana* (By Felipe Guaman Poma de Ayala (https://commons.wikimedia.org/wiki/File:Yupana.jpg)

The Chinese 算盘 (*suàn pán*) and the Japanese そろばん (*soroban*) have a similar structure but with a different number of beads (Sun 2015; see also Chap. 5).

The Russian счёты (*schoty*) is not organised in columns but in rows, where one bead is equivalent to the bead of the adjacent row below, with one exception (the row for quarter kopek, an ancient coin) (Figs. 9.9, 9.10, 9.11 and 9.12).

Inspired by the Russian abacus, the so-called Slavonic abacus (or *arithmetic rack*) was introduced in Europe by Kempinsky (1921) who gave it the name of *Russische Rechenmaschine* (Rottmann and Peter-Koop 2015); see also Sect. 9.2.3.

Yet, during time, the positional value, where rows represent units, tens, hundreds, thousands and so on, was replaced by the convention that each bead represents a unit, whichever is the column or row (see also Sect. 9.4.1) (Fig. 9.13).

9.2.2.3 Artefacts for Multiplication

Pithy tables (*or nine times tables or multiplication tables*) are popular all over the world with different names. For instance, in Italy, the table of Fig. 9.14, printed in the last page of notebooks until 50 years ago, was named 'tavola pitagorica', but it is not clear why Pythagoras is mentioned.

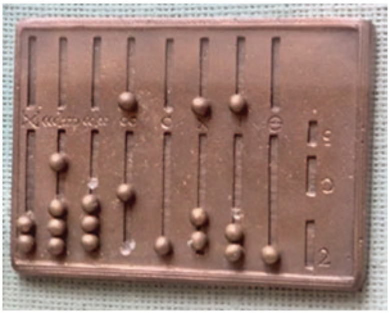


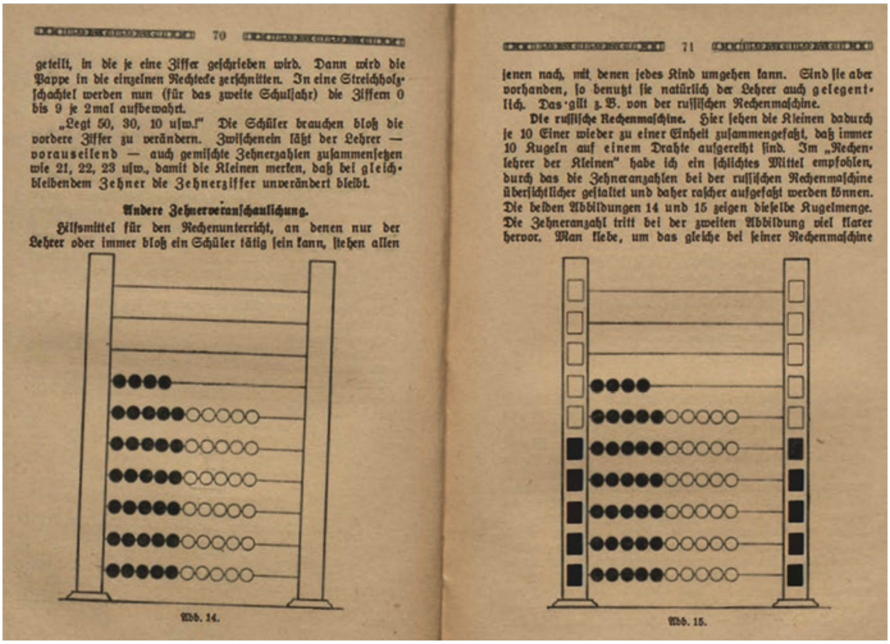
Fig. 9.9 Roman abacus (https://commons.wikimedia.org/wiki/File:RomanAbacusRecon.jpg)

Fig 9.10 A precious ancient jade *suàn pán* 算盘 (personal collection of the first author)

Fig 9.11 A *soroban* (personal collection of the first author)

Fig 9.12 A *schoty* (personal collection of the first author)

According to historians (Lam and Ang 2004, p. 73 ff.) in China, this table was an integral part in the rudiments of counting, as from the seventh century BCE (this volume, Sect. 15.5). Later, it was known as the 'nine nines song', as the learner had to recite the numbers in a singing manner to memorise the table. The reduced forms in Chinese textbooks (Fig. 9.15) draw on commutative property (Cao et al. 2015).



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geteilt, in die je eine Ziffer geschrieben wird. Dann wird die Pappe in die einzelnen Rechtecke zerschnitten. In eine Streichholzschachtel werden nun (für das zweite Schuljahr) die Ziffern 0 bis 9 je 2mal aufbewahrt.

„Legt 50, 30, 10 usw.!" Die Schüler brauchen bloß die vordere Ziffer zu verändern. Zwischenein läßt der Lehrer — vorauseilend — auch gemischte Zehnerzahlen zusammensetzen wie 21, 22, 23 usw., damit die Kleinen merken, daß bei gleichbleibendem Zehner die Zehnerziffer unverändert bleibt.

Andere Zehnerveranschaulichung.

Hilfsmittel für den Rechenunterricht, an denen nur der Lehrer oder immer bloß ein Schüler tätig sein kann, stehen allen

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jenen nach, mit denen jedes Kind umgehen kann. Sind sie aber vorhanden, so benutzt sie natürlich der Lehrer auch gelegentlich. Das gilt z. B. von der russischen Rechenmaschine.

Die russische Rechenmaschine. Hier sehen die Kleinen dadurch je 10 Einer wieder zu einer Einheit zusammengefaßt, daß immer 10 Kugeln auf einem Drahte aufgereiht sind. Im „Rechenlehrer der Kleinen" habe ich ein schlichtes Mittel empfohlen, durch das die Zehneranzahlen bei der russischen Rechenmaschine übersichtlicher gestaltet und daher rascher aufgefaßt werden können. Die beiden Abbildungen 14 und 15 zeigen dieselbe Kugelmenge. Die Zehneranzahl tritt bei der zweiten Abbildung viel klarer hervor. Man klebe, um das gleiche bei seiner Rechenmaschine

Fig. 9.13 The arithmetic rack (Kempinsky 1921)

Fig. 9.14 'Tavola pitagorica' from an Italian notebook (around 1960, personal collection of the first author)

TAVOLA PITAGORICA

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

一一得一 （1×1=1）								
一二得二 （1×2=2）	二二得四 （2×2=4）							
一三得三 （1×3=3）	二三得六 （2×3=6）	三三得九 （3×3=9）						
一四得四 （1×4=4）	二四得八 （2×4=8）	三四十二 （3×4=12）	四四十六 （4×4=16）					
一五得五 （1×5=5）	二五一十 （2×5=10）	三五十五 （3×5=15）	四五二十 （4×5=20）	五五二十五 （5×5=25）				
一六得六 （1×6=6）	二六十二 （2×6=12）	三六十八 （3×6=18）	四六二十四 （4×6=24）	五六三十 （5×6=30）	六六三十六 （6×6=36）			
一七得七 （1×7=7）	二七十四 （2×7=14）	三七二十一 （3×7=21）	四七二十八 （4×7=28）	五七三十五 （5×7=35）	六七四十二 （6×7=42）	七七四十九 （7×7=49）		
一八得八 （1×8=8）	二八十六 （2×8=16）	三八二十四 （3×8=24）	四八三十二 （4×8=32）	五八四十 （5×8=40）	六八四十八 （6×8=48）	七八五十六 （7×8=56）	八八六十四 （8×8=64）	
一九得九 （1×9=9）	二九十八 （2×9=18）	三九二十七 （3×9=27）	四九三十六 （4×9=36）	五九四十五 （5×9=45）	六九五十四 （6×9=54）	七九六十三 （7×9=63）	八九七十二 （8×9=72）	九九八十一 （9×9=81）

Fig. 9.15 The reduced pithy table from an old Chinese textbook

In each line, only the case of $a < b$ for $a \times b$ is written.

An original way of approaching the multiplication table is suggested by Baccaglini-Frank et al. (2014) within the *PerContare* project, drawing on Pythagoras' studies on figurate numbers and Euclid's further development of geometric algebra (this volume, Chap. 7). The product of two numbers $a \times b$ is represented in the table by a rectangle with sizes a and b. In this way, the construction of the table is justified by a spatial approach, and some properties of multiplication (e.g. commutative, distributive) are evident.

Napier's bones (*or rods*) are an artefact for multiplication, drawing on nine times table. Each rod is a strip of wood, metal or heavy cardboard. A rod's surface comprises ten squares: the first holds a single digit, while the others comprise two halves divided by a diagonal line. In each square there are the multiples of the number on the top. In Fig. 9.16, there is a collection of Napier's rods together with an example of application.

A similar approach, on paper and pencil without rods, is in the *Gelosia (or lattice) multiplication* (Siu 2015; see Fig. 9.17). It is an algorithm, probably from Arabic culture, but later spread also in Europe (through Italy): the advantage of this method in the classroom is that every result from the nine times table is written and only later combined with others. Hence, the control of the single steps of the process is fostered.

9.2.2.4 Number Line

The *number line* (Bartolini Bussi 2015, and this volume, Chaps. 15 and 19) draws on the Euclidean tradition of representing numbers with line segments. It was transformed into a teaching aid in Europe in the seventeenth century. Now number lines are part of everyday experience of pupils, either in games (e.g. the board *game of the Goose* especially popular in Southern Europe) or in everyday tools (e.g. the graded ruler or scales in measuring instruments with direct reading).

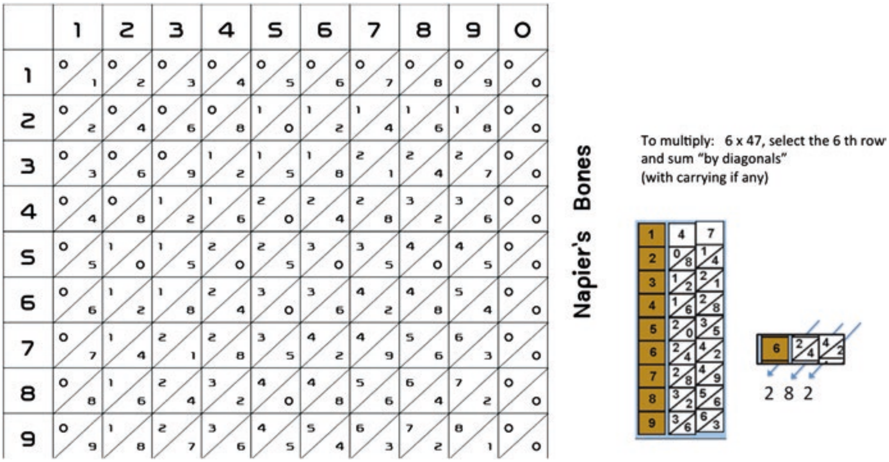

Fig. 9.16 *Left*: Napier's bones. *Right*: instruction

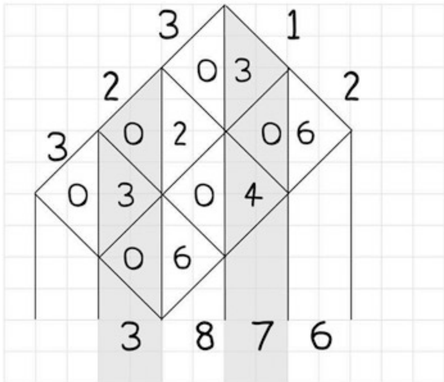

Fig. 9.17 Gelosia multiplication: 323 × 12 = 3876

9.2.2.5 Songs, Poems and Dance

The recitation of nine times table as a song was already mentioned above. The tradition of recitation in mathematics learning was widespread in many parts of the world too. For instance, in India (Karp and Schubring 2014):

> The recitation involved knowing how to chant Vedic verses following many systematic combinations of their syllables, first in order, then inverting one verse/syllable after another, then reciting it backwards, and so on, so that the recitation itself could be seen as an application of a systematical "mathematical" combination". (p. 71)

There are also cultures in Africa where dancing and singing together is a way to recite and learn numbers (Electronic Supplementary Material: Sarungi 2017; see also Zaslavsky 1973, Chap. 10).

Meaney, Trinick and Fairhall (2012) report an activity about whole number arithmetic, taken from a New Zealand television programme and based on the principles of *kapa haka*, a traditional team dance where actions emphasise the sung or chanted words, using the body as the instrument for delivery.

The role of the body is evident also in a video clip (Electronic Supplementary Material: Arzarello 2017) about a project developed in an Italian first-grade classroom (this volume, Chap. 15). Students learn to recite the numbers according to the regular and transparent Chinese structure that is different from the Italian one. Hence, they say: ‘nine, ten, ten-one, ten-two, ten-three…’ accompanying this recitation with large-size arm gestures, which help them to keep up the pace.

9.2.2.6 Games

Many games embody number properties. Some examples follow.

- Traditional games (e.g. the goose game; see above); *mancala* (Fig. 9.18), an African game with seeds (Zaslavsky 1973, Chap. 11).
- Magic squares (in China, Africa, Europe) (Fig. 9.19).
- Playing cards with special patterns fostering *subitising* (see Sect. 7.2.1).
- Games from recreational mathematics.

Recreational mathematics has been popular all over the world, since ancient times (see also Zaslavsky 1973, Sect. 9.4). Singmaster has collected a large set of sources, many of which concern WNA.[1] Gardner has published hundreds of columns in Scientific American and other books.[2] Famous collections have been published in the former Soviet Union (Kordemsky 1992) and in Latin America. The book by Malba Tahan (1996), the pen name of Júlio César de Mello e Souza, tells the fictitious story of an Arab mathematician of the fourteenth century, as a series of

Fig. 9.18 A *mancala* (personal collection of the first author)

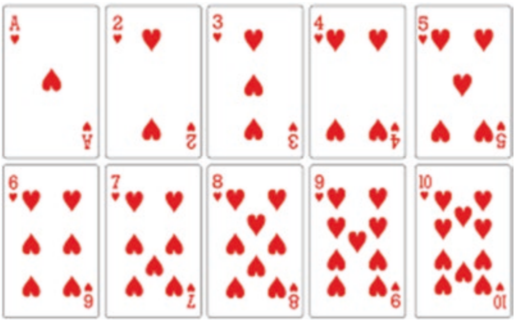

Fig. 9.19 Playing cards (personal collection of the first author)

[1] http://www.puzzlemuseum.com/singma/singma-index.htm

[2] http://martin-gardner.org

tales in the style of the Arabian Nights, but revolving around mathematical puzzles. The book was very popular in Brazil and was translated into many languages including Arabic.

Games have been implemented also in devices of computer technology, as computer games (Bakker et al. 2015) and applets for multitouch technologies (see Sect. 9.3.4).

9.2.2.7 Everyday Artefacts

Everyday artefacts reflect mathematical ideas. Some examples:

- Banknotes and coins.
- Cake boxes (like the ones used in the Hou Kong School, this volume Chap. 11) or egg boxes with regular organisation of places.
- Stamp sheets (organised in ten lines of ten stamps each) (see Inprasitha 2015).

9.2.2.8 Textbooks

Mathematics textbooks have been for centuries the most widespread artefacts all over the world. In this chapter, we wish to put only a signpost for this issue that is considered elsewhere in this volume (Chap. 11). We shall devote some space to d-book only (see Sect. 9.3.4.4).

9.3 When Artefacts Are Teaching Aids: The Construction of Mathematical Meanings

A cultural artefact may become a teaching aid when it is used in schools with didactical intentions. In the previous section, we have collected a gallery of examples, most coming from the history of mathematics. We have also briefly mentioned some didactical use of them. In this section, we shall deepen this point, discussing some issues relating to the teaching and learning process.

9.3.1 Some Modern Artefacts

A teacher or a mathematics educator may design an original artefact with specific intentions. Some examples from the history of mathematical instruction follow.

9.3.1.1 Cuisenaire Rods

Cuisenaire rods (this volume, Chaps. 8 and 10) are used to represent numbers with coloured rods of different length, in the trend already introduced by Froebel and Montessori. They were designed in the 1920s by the wife of Georges Cuisenaire, a Belgian educator, in order to make arithmetic visible. Some decades later Caleb Gattegno named them Cuisenaire rods and started to popularise them. In this case, the number is approached through measuring. Nesher (this volume, Chap. 8) analyses them according to Peano's axioms. These rods may be used also to create challenging tasks (Ball and Bass 2015). They have also inspired some apps like number bonds by Diana Laurillard.[3]

9.3.1.2 Multibase Arithmetic Blocks

MAB (*multibase arithmetic blocks*) are one of the most popular teaching aids to introduce place value (see Chap. 4). They provide concrete representations for the number bases (Dienes 1963). They model numbers with objects hinting at different dimensions (see Ladel and Kortenkamp 2015; Rottmann and Peter-Koop 2015). In spite of the diffusion, they have been criticised from an epistemological and cognitive perspective (Stacey et al. 2001), suggesting, instead, of using *Linear Arithmetic Blocks (LAB)* that model numbers with length, showing the position of numbers on a number line.

9.3.1.3 Spike Abacus

The *spike abacus* is inspired by the abaci of the past containing different wires with beads referring to units, tens, hundreds and similar (Fig. 9.20). In some traditions (e.g. Baldin et al. 2015), it is common to use different colours for units, tens and so on and to transcribe multi-digit numbers using pens of different colours, with the purpose of making differences more evident. This choice seems not advisable, as attention is focused on colours and exchange conventions rather than on order and position.

9.3.2 Artefacts for Place Value: The Cultural Roots of Epistemological Obstacles

Some of the examples of our gallery address a crucial issue in WNA, which is place value (this volume, Chaps. 3 and 5). Counting rods (筹; *chóu*) and all the kinds of abaci, for instance, are strongly related to place value. As argued in this volume

[3] http://thinkout.se/thinkout-products/number-bonds/

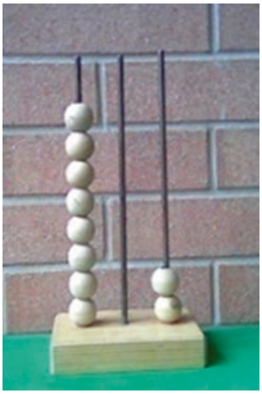

Fig. 9.20 A monochrome spike abacus (personal collection of the first author)

(Chap. 5), place value, at least in the Western culture, hints at an *epistemological obstacle*, with influences on the teaching and learning processes.

An epistemological obstacle may be described, after Brousseau and Bachelard, as follows:

> Brousseau's approach is based on the assumption that knowledge exists and makes sense only because it represents an optimal solution in a system of constraints. […] In Brousseau's view knowledge is not a state of the mind; it is a solution to a problem, independent of the solving subject. (Fauvel and van Maanen 2000, p. 162)

Usually epistemological obstacles are related to the historical process of constructing mathematical knowledge by mankind and are likely to appear anew in the mathematics classrooms. But, as we show below, this idea has to be carefully analysed with a lens of cultural and language relativism. The cultural roots of epistemological obstacles have been discussed by Sierpinska (1996) and Radford (1997; see also D'Amore et al. 2016). In a recent study on language impact on the learning of mathematics, Dong-Joong et al. (2012) observed:

> More generally, this study brings to the fore the importance of teachers' awareness of the unique, language-dependent properties of the discourse to which they are going to usher their students. The teachers need to be cognizant of those language-specific features of the discourse that may support learning and of those that may hinder successful participation. Thus, to support meaningful learning in English-speaking classes, instructors may wish to deliberately capitalize on the existing lexical ties between students' informal talk and formal mathematical discourse on infinity. But these teachers should also remember that the continuity has its dark side, in that it may hinder the necessary change: at different levels, the same words are used in different ways, but the required transformation may be difficult for the students not just to implement, but even to see. (p. 106)

The study was about infinity with secondary school students in the USA and Korea, but the same observation might be applied to place value in WNA. From a Western perspective, we know (Menninger 1969, p. 39 ff) that the early representations of whole numbers were in most cases based on additive rules (this volume Chap. 5). Representation of numbers and computing seemed to be quite different issues: numbers were represented in additive form, and the operation to solve arith-

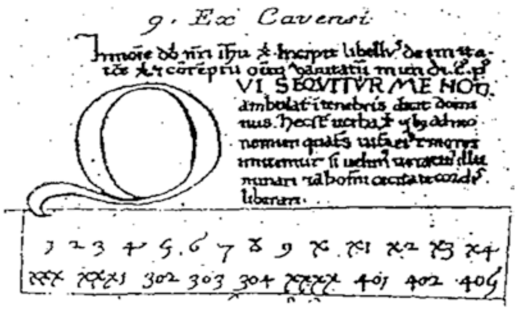

Fig. 9.21 An ancient Italian manuscript

metic problems were solved by some artefacts before transcribing the results. In the previous section, we have mentioned the Roman abacus, which worked on a positional rule, while numbers were written drawing on additional system.

The situation was different in China (this volume, Chaps. 3 and 5), where the links between the representation of whole numbers (in both words and symbols) and the artefacts for computing (e.g. counting rods and *suàn pán*) was very strict from the beginning and with no interruption between them.

In Europe, when representation of numbers according to non-positional (additive) systems was in use, no need for ‘zero’ to represent the empty positions on the abacus emerged. When eventually ‘zero’ was introduced in Europe from the East through the Arabic tradition together with the so-called Hindu-Arabic digits, the advantage of this introduction was not immediately acknowledged.

The medieval Italian manuscript of Fig. 9.21 shows the problems in shifting from the Roman notation to the new one (Fauvel and van Maanen 2000, p. 151).

What happens today in the (Western) mathematics classrooms?

> When 7-year-old students are asked to write numbers, a common mistake in transcoding from number words to Hindu-Arabic numerals shows up: some students write ‘10,013’ instead of ‘113’ as the zeroes on the right (100) are not overwritten by tens and units. […] This mistake is stable and resists direct teaching of place value. […] rather than using place value conventions, the students seem to use digits to transcribe oral numerals. (Bartolini Bussi 2011, p. 94)

The (Western) epistemological obstacle requires recourse to specific activities to be overcome. It is necessary to reconstruct the link between representing numbers orally and in written forms, which seems so natural in Chinese classrooms (this volume, Chap. 15). This may be done using some kind of artefacts and tasks. For instance, it is possible to use cultural artefacts such as abaci or counting rods (inherited from the history of mathematics and strictly linked to place value development) or artefacts from everyday life. This last case is reported in a study by Young-Loveridge and Bicknell (2015), who assert that ‘place value understanding is inherently multiplicative’ and that ‘multiplicative thinking involves working with two variables (number of groups and number of items per group) and these are in a fixed ratio to each other, in a many-to-one relationship’ (p. 379). This implies that ‘a key feature of place-value development is the shift from a unitary (by ones) way of thinking about numbers to a multi-unit conception, e.g., tens and ones’ (p. 381). Drawing

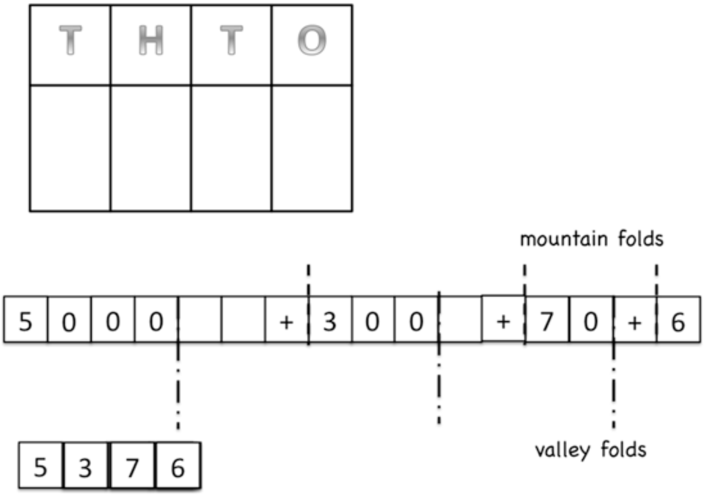

Fig. 9.22 Place value chart and a foldable strip representing 5376; folded it shows the four-digit number; unfolded it shows the sum of thousands, hundreds, tens and ones

on these results, the authors have designed and carried out a study with 35 5-year-old children who solved word problems including multiplication, division and place value. Place value was approached by using everyday artefacts (e.g. egg carton with exactly ten eggs; gloves with five fingers). The study showed that most children were able to work with fives and tens by the end of the programme.

In both cases, artefacts become teaching aids: in the former case, they are taken from the history of culture; in the second case, they are taken from everyday life. The choice depends on the implicit systems of values and on the image of mathematics: to construct the cultural identity by referring to the (local) history of mathematics or to highlight the links between arithmetic and everyday life. The difference is not so strict, as in some cases, the use of cultural artefacts from the history of mathematics is still extant in everyday life, as Chinese 算盘 (*suàn pán)* and Japanese そろばん (*soroban*).

Other very simple artefacts may be used, like *place value charts* or *foldable strips*. A foldable strip shows the number as a sum of thousands, hundreds, tens and units (when opened) and as a four-digit number when folded as the zeroes are overwritten (Fig. 9.22).

9.3.3 Artefacts for Low Achievers: Another Example of Cultural Difference

We have mentioned the number line, a cultural artefact, whose history in Europe may be traced back to the importance of geometry since the classical age (Bartolini Bussi 2015). The number line is very often used in the Western mathematics

Fig. 9.23 A child jumping on a floor number line

education to introduce addition and subtractions by motion forwards and backwards, but it is not so popular in China (this volume, Chap. 15).

Figure 9.23 is taken from a video clip (Electronic Supplementary Material: Bartolini Bussi 2017), where a student jumps on the floor, exploring a big-size number line.

A smaller-size number line drawn on a sheet of paper (Bartolini Bussi 2015) may be used with low achievers (e.g. dyscalculic children) to introduce addition and subtraction. The following is the prototype of a dialogue (one-to-one interaction) between a low achiever and a caregiver. The child can read numbers but cannot retrieve from memory simple arithmetic facts. There is a number line drawn as a linear sequence of positions numbered from 0 to 10. The pawn to be moved is called Tweety. The task is to calculate 4 + 3.

Adult: 'Put Tweety on the 4.'
(done)
Adult: 'Keep Tweety steady and count on 3 with your finger.'
(done)
Adult: 'Read the number.'
Child: 'Seven.'
Adult: 'Good job! 4 + 3 = 7.'

The activity aims at constructing a very simple procedure to be used by the low achiever first in a guided way and then independently, to acquire autonomy in the construction of simple number facts (addition in this case). The signs + and – on the top are reminders for the direction for addition and for subtraction (Fig. 9.24).

We may compare this activity with the more common activity of tracing small arches on the number line, as in Fig. 9.25.

Teachers report difficulties with low achievers as they are not able to coordinate counting with tracing small arches: sometimes they count twice the vertical segments pointing at each number (both up and down) and become confused.

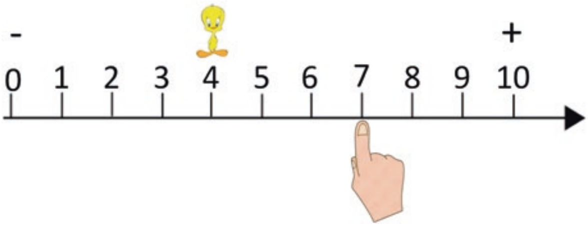

Fig. 9.24 Moving Tweety on the number line

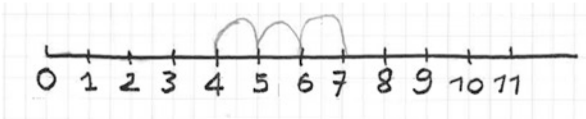

Fig. 9.25 Drawn arches on the number line

9.3.4 Artefacts and Mathematical Meanings

An artefact is never neutral. It always ‘contains’ the aims and the knowledge of its designers. This is true for both the artefacts coming from the history of mathematics (whose designers are sometimes lost in the mists of time) and the modern artefacts, designed with specific didactical intentions. The design of an artefact depends on the designers’ background knowledge about the mathematics involved and on the intentions about what to mediate. Later, the teachers’ background and intention will determine the classroom use.

9.3.4.1 Strings with Beads and Arithmetic Racks

In many pre-schools in Italy, there is a custom/tradition to hang strings with moving beads (similar to Greek worry beads) on the wall to trace the passing of time (e.g. seven beads for a week) or the situation of present/absent pupils (e.g. 28 beads for the whole class). The number of beads on the string depends on the context: 7 beads for a week, 28 beads for the class and so on. The manipulation reminds one of the manipulation of an arithmetic rack (Slavonic abacus) where beads are dragged for counting. But there is a difference: the strings with beads are dependent on the context (7 days in a week; 28 pupils in a class), while the arithmetic rack is decontextualised and can be used to count every small collection. Rather the number of beads depends on the choices made in mathematics, to use base ten to count. In other words, it is a cultural artefact, where culture is here referring to mathematics culture. Even before being introduced to place value, pupils use a rack with ten beads in every line (see Sect. 9.4.1.1). Hence, they may practise counting and notice that when you go to the next row, some kind of language regularity happens: 21, 22, …, 31, 32 and so on. This approach does not require the convention of substituting one bead with ten beads on another wire (as in other abaci; see Sect. 9.2.2.2) as each bead is a unit. In other words, the collection of beads is similar to tallies (Sect. 9.2.2.1).

9.3.4.2 Artefacts and the Learner's Processes

Each artefact drives the actions of the user and is driven by the user in a coextensive process. This coextensive process changes the users' thinking. This is consistent with the quotation from Vygotsky in Sect. 9.2.1. In that way the design of an artefact influences the way the student will use it, the knowledge a student will learn and internalise while working with the artefact and also the student's image of mathematics. In the process, not only the design of an artefact but also the task and the acting influence the student's processes (see Sect. 9.3).

The arithmetic rack may be improved, with particular mathematical aims. A designer who knows that we are usually able to recognise numbers up to five simultaneously and larger numbers only in a quasi-simultaneous way can colour the balls of the arithmetic rack with a structure of five and also add black and white labels (Fig. 9.13). Such design improvements have been refined over decades and are still visible in some modern artefacts.

The exclusive use of an artefact as an 'object on its own' could lead to sticking with direct modelling activities and using only counting strategies instead of using structural features of an artefact (like a structure by fives and tens) and developing more sophisticated mental calculation strategies.

We use the arithmetic rack as an example. It should support children to replace counting as an arithmetic strategy with more advanced strategies. Children can continue to slide the balls one by one, still counting, whereas it is possible to move several balls at once. The children have to understand at least implicitly what the artefact was made for and to try to follow the intentions of whoever created the artefact. These assumptions about the designer's intentions play an important role in how they should use an artefact, and it is the responsibility of the teacher to guide students in the appropriate use. Therefore, it is essential that the teacher is highly competent in mathematics and mathematics education. He or she is responsible for the right and good choice of an artefact, and is the one who has to show the children how to work with it. To use an artefact in a constructive way, it is necessary for a child to become familiar with the artefact and its structure (see the example of the giant Slavonic abacus for pre-school in Sect. 9.4.1.1). The teacher's instruction is needed (at least for some children) to support the development of mathematical meanings and strategies; so teachers have a role as cultural mediators with respect to mathematical content.

9.3.4.3 From Concrete to Thought Experiment

Whereas 'smaller' numbers (in particular numbers up to 100) can easily be presented as sets of items by artefacts or physical objects, the situation fundamentally changes with bigger numbers (like 123,456). With the extension of the number range, artefacts become less and less important as concrete representations of numbers. Instead, mental 'enlargements' of artefacts are used frequently. For instance, how would one million look like if we use base-ten MAB (Schipper et al. 2000)?

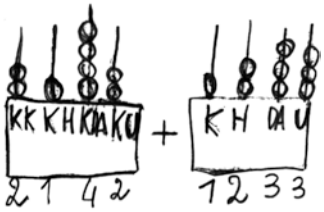

Fig. 9.26 A second grader draws two exemplars of spike abacus close to each other to represent an eight-digit number by juxtaposing them

Table 9.1 Four-Phases_Model to support the development of basic computational ideas

Phase 1	Concrete usage of manipulatives and verbalisation of operations
	Teacher and child actively use the material and verbally describe their operations and their meaning. When the child is confident in working with the material, the child takes over and verbalises the operation itself.
Phase 2	Verbal description of the imaginative use of the manipulative in sight
	With the manipulative in sight, the child describes the operations on the manipulative to the teacher or a fellow student who performs the according operations following the child's descriptions.
Phase 3	Verbal description of the imaginative use of the covered manipulative
	With the manipulative covered by a screen/shield, the child describes the operations on the manipulative to the teacher or a fellow student who performs the according operations following the child's descriptions.
Phase 4	Verbal description of the mental operation
	The child verbally describes the operations without the manipulative being present in any form other than the child's imagination. The tasks are given in a symbolic representation

In this case, artefacts tend ase tend less to be used as manipulatives and auxiliaries for solving calculation problems but rather are used as reference points for English (1997). So, for instance, the steps on the number line are equal for 58 + 37 and 12,358 + 37. The wires in a spike abacus may be duplicated to reach thousands and millions (see the representation of a second grader in Fig. 9.26).

A successful teaching and learning process depends on the ability to focus upon the 'relevant' aspects of the actions and thereby to link the symbolic representation and the enactive representation (in terms of concrete actions on manipulatives).

A specific *Four-Phases-Model* (see the Table 9.1) to assist the gradual shift from material to mental images was reported by Rottmann and Peter-Koop (2015; cf. Wartha and Schulz 2012) to support the development of mathematical concepts and mental strategies especially for students that experience severe learning difficulties, assisting such internalisation processes. Mental images and representations should increasingly replace concrete actions on manipulatives over time, although working at any single point usually still involves moving between concrete and mental images (Roberts 2015).

This model is based on initial ideas of Bruner and the further development of Bruner's theory by the Swiss psychologist Aebli (1976). Bruner (1973) distinguished three types of representational systems: the enactive, the iconic and the symbolic representation. While the enactive representation is based upon actions, the iconic representation comprises both, pictures and mental images. The symbolic representation involves mathematical symbols (as written numbers or operation symbols) as well as language. Bruner strongly links learn-

> ing processes to translations of one representational system into another. And Aebli in addition describes gradual internalization processes from enactive to mental actions, which focus on the transition from one representation to another.
>
> With emphasis on verbal descriptions of enactive and mental actions, the Four-Phases-Model stresses the relevance to assist the development of mental images by a gradual and systematic removal of the manipulatives. (Rottmann and Peter-Koop 2015, p. 366)

This Four-Phases-Model acknowledges the need for verbal descriptions when using concrete manipulatives and a transitional phase from manipulating with material to mental operations that activate a mental concept which allows the child to imagine the actions required in order to solve an addition or subtraction problem (Electronic Supplementary Material: Rottmann 2017).

9.3.4.4 Not Only Counting

The examples above are mainly based on counting, although there is an intentional shift later towards mental strategies. According to Zhou and Peverly (2005), 'Overreliance on counting strategies at this age will hinder children's development of abstract mathematical reasoning abilities' (p. 265). Surely counting risks Putting measuring into the shade (see this volume, Chaps. 13 and 19), but this is not the only possible risk.

An interesting approach in Chinese kindergartens and early primary classrooms was reported (Ni et al. 2010; Cheng 2012). Children are given a multiple-classification task with sets of 2,3,4,5,6,7,8,9 small faces. For instance, children are shown four faces and asked to identify attributes that may be used to classify the faces into different groups. These four faces feature three attributes: one face has a hat and three do not; three happy faces and one angry face; two yellow faces and two red faces.

Teachers ask their students to observe and analyse the attributes and relationships of the four faces. Then, they guide the students' use of black and white beads to model the relationships as they solve addition and subtraction problems within this universe of 4 (e.g. $1 + 3 = 4$, $3 + 1 = 4$, $2 + 2 = 4$; $4 - 1 = 3$, $4 - 3 = 1$, $4 - 2 = 2$). Next, the students develop their understanding of part-whole relations for the other numbers (from 2 to 10), by doing classification tasks with these numbers. In this way, children are led to practise decomposition in additions and subtractions. They makes notes on a 10×10 grid.

This example refers to combinatorial thinking and reasoning (this volume, Chap. 13). The use of a 10×10 grid is consistent with the pattern and structure approach as discussed in this volume (Chap. 7).

The activity of composition/decomposition is described also by Inprasitha (2015) as shown in Fig. 9.27. How many? is an activity for learning decomposition for first grade students. They drop the balls in the box and guess '*How many are there?*', '*How many are hidden?*'. Then they fill the number of balls on the card with the correct number. For example, drop five balls in the box, and fill two and three balls on the card (Inprasitha 2015). It is also consistent with the activity for first graders practised in China (Fig. 9.28).

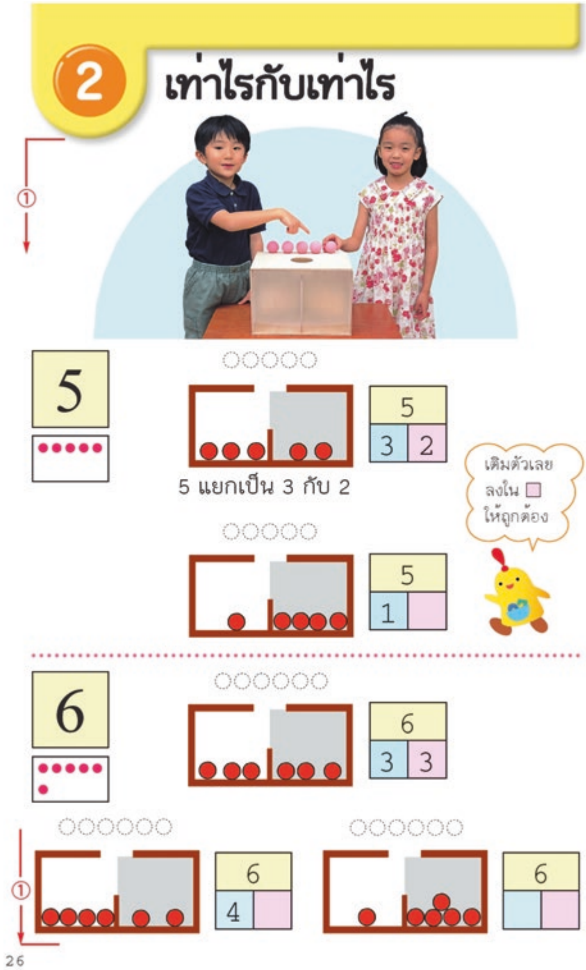

Fig. 9.27 The drop-box game extracted from Inprasitha and Isoda (2010)

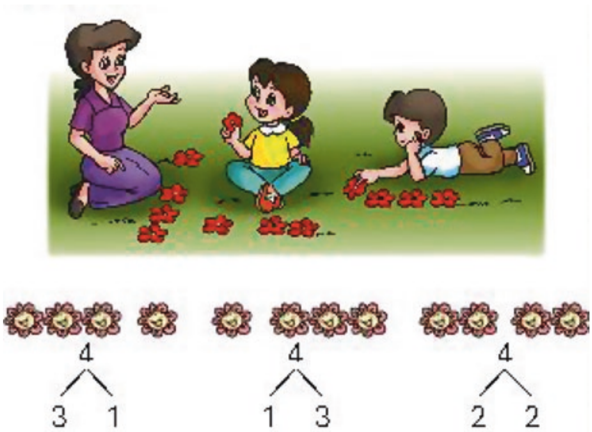

Fig. 9.28 Activity for first graders (a Chinese textbook)

9.3.5 Concrete and Virtual Manipulatives

9.3.5.1 A Possible Contrast

The previous sections have introduced different examples of artefacts, with a special focus on those that are concretely manipulable. In the last decades, the plentiful supply of technologies has fostered the production of virtual manipulatives. The contrast between concrete and virtual artefacts has been addressed in 2009 by Sarama and Clements (2009), who analysed several studies. Their conclusion is that the contrast is not between concrete and virtual manipulatives:

> Manipulatives are meaningful for learning only *with respect to learners' activities and thinking*. Physical and computer manipulatives can be useful, but they will be more so when used in comprehensive, well planned, instructional settings. Their physicality is not important – their manipulability and meaningfulness make them educationally effective. In addition, some studies suggest that computer manipulatives can encourage students to make their knowledge explicit […] but rigorous causal studies have not been conducted to our knowledge. Such research, using randomized control trials, must be conducted to investigate the specific contributions of physical and computer manipulatives to particular aspects of mathematics teaching and learning. (pp. 149–50)

In ICMI Study 17 (Hoyles and Lagrange 2010), only one example of a project for primary school with technologies is described, i.e. the *SYL Project* (Sketchpad for Young Learners, p. 66). A task is reported, i.e. the jump-along activity, for grades 3–5, with jumps along a basic number line on the screen, where students choose different parameters for the number of the jumps and the size of each jump. This project aims at supporting the existing curriculum, reifying the existing teaching practice on the number line.

With a similar aim, a library of virtual manipulatives was created in the USA.[4] On the home page of the NLVM website, one reads:

> The National Library of Virtual Manipulatives (NLVM) is an NSF-supported project that began in 1999 to develop a library of uniquely interactive, web-based virtual manipulatives or concept tutorials, mostly in the form of Java applets, for mathematics instruction (K-12 emphasis). […] Learning and understanding mathematics, at every level, requires student engagement. Mathematics is not, as has been said, a spectator sport. Too much of current instruction fails to actively involve students. One way to address the problem is through the use of manipulatives, physical objects that help students visualize relationships and applications. We can now use computers to create virtual learning environments to address the same goals.

The view that a virtual manipulative may address the 'same goal' as concrete manipulatives may be contentious first, concrete manipulation may be different from mediated manipulation (e.g. by means of a mouse); second, the facilities offered by technologies may allow to produce manipulatives which are not simulations of the concrete ones, but completely new artefacts instead, with their own intentional designs. Some examples follow.

[4] http://nlvm.usu.edu

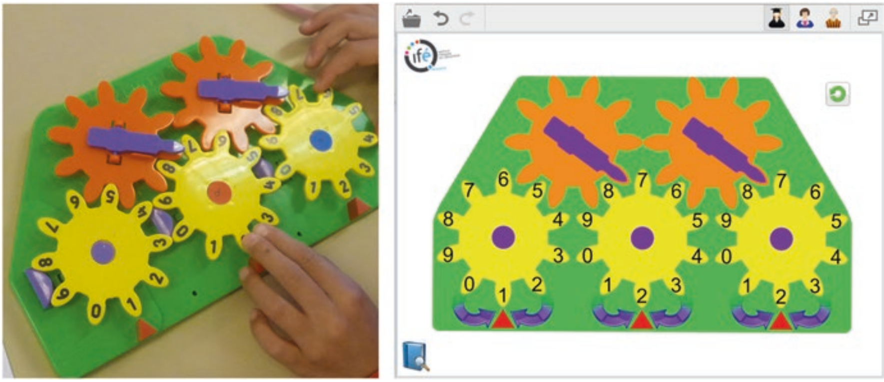

Fig. 9.29 The pascaline (*left*) and the e-pascaline in a Cabri Elem e-book (*right*), both displaying number 122

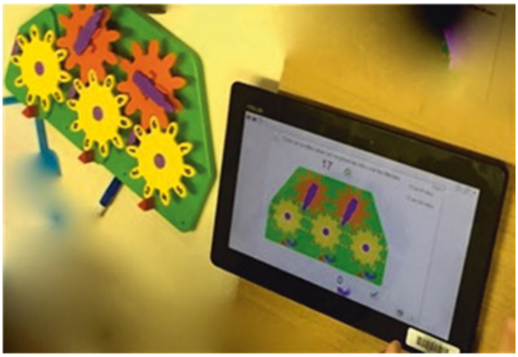

Fig. 9.30 The tablet version of the e-pascaline

9.3.5.2 The Pascaline and the e-Pascaline

Soury-Lavergne and Maschietto (2015) reported an international experiment, carried out in France and Italy, with a pair of artefacts: a concrete one (pascaline) and a virtual one (e-pascaline). The pascaline (Fig. 9.29, left) is an arithmetic machine made of gears, named after the historical machine of Blaise Pascal, while the e-pascaline (Fig. 9.29, right), developed with the Cabri Elem technology,[5] is close enough to the pascaline to enable students to transfer some schemes of use (Fig. 9.30).

The pascaline displays three-digit numbers and enables arithmetic operations to be performed. Each of its five wheels has ten teeth and can rotate in two directions. Teeth of the three lower wheels have digits from 0 to 9 and display units, tens and hundreds from the right to the left. The upper wheels automatically drag the lower wheels when needed for place-value notation. The jerky motion of the wheels, rotat-

[5] http://educmath.ens-lyon.fr/Educmath/recherche/equipes-associees-13-14/mallette/cabri-elem-logiciels

ing one tooth at a time, mediates the recursive approach to number, adding or subtracting one unit according to the clockwise or anticlockwise direction. It also links addition and subtraction as inverse operations.

The e-pascaline is close enough to the pascaline to enable students to transfer some schemes of use. Yet it is also different as some procedures with the physical pascaline may be inhibited with the e-pascaline (details in Soury-Lavergne and Maschietto 2015). For instance, the e-pascaline aims to provoke the evolution from the iteration procedure to the decomposition procedure in addition. The two artefacts are combined to help students to overcome some of their limitations and to offer the possibility of a rich experience leading to a flexible understanding of mathematical notions. The e-pascaline may work also on tablets. Using the e-pascaline on tablets (Fig. 9.30), with a touchscreen, gives students a more direct access to the action on the wheels even through the use of the action arrows (Soury-Lavergne and Maschietto 2015). This issue leads us to the further section where multitouch technologies are addressed.

9.3.5.3 Multitouch Technologies

Multitouch technologies introduce new possibility in the design of virtual artefacts. According to Sinclair and Baccaglini-Frank (2015):

> With multitouch technology, the interaction becomes more immediate, as the fingers contact the screen directly, either through tapping or a wide variety of gestures. Further, the screen can be touched by multiple users simultaneously at the same time, which invites different types of activity structures than the computer or laptop. [The authors refer to the extended neuroscientific literature pointing to the importance of fingers in the development of number sense and continue as follows.] Basic component abilities that can be powerfully mediated through multi-touch technology are: 1) subitising; 2) one-to-one correspondence between numerosities in analogical form and fingers placed on screen/raised simultaneously/counting with fingers, and in general finger gnosis; 3) fine motor abilities; and, 4) the part-whole concept. (p. 670)

Some examples below indicate the potentialities of multitouch technologies.

TouchCounts is an iPad app (designed by Nathalie Sinclair) where children use their fingers, eyes and ears to learn to count, add and subtract. By using simple gestures to create and manipulate their own numbers, children develop a strong number sense at least for some early steps. TouchCounts is aimed at the use of fingers in order to positively affect the formation of number sense and thus also the development of calculation skills. There are two sub-applications in TouchCounts, one for counting (1, 2, 3, …) and the other for operations (addition and subtraction). In the former the first tap produces a disc containing the numeral '1'. Subsequent taps produce sequentially numbered discs. In the latter, children create arbitrary whole numbers and explore basic number operation concepts by pushing (squeezing) numbers together (into new, larger numbers) or by splitting numbers apart (into new, smaller numbers). The strong relationship between fingers and numbers has the potential to address the issue of *finger gnosis* (literally 'finger knowledge'),

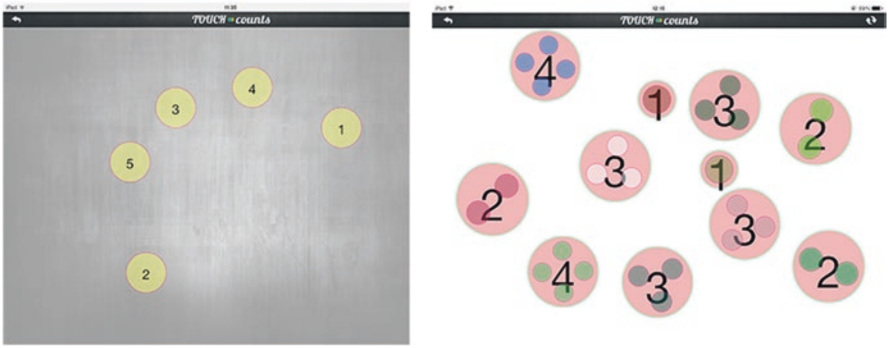

Fig. 9.31 TouchCounts

defined as the ability to differentiate one's own fingers without any visual clues when they are touched (Sinclair and Pimm 2014) (Fig. 9.31).

For *subitising* (Sect. 7.1.1), Baccaglini-Frank and Maracci (2015) analyse other apps: *Ladybug Count*[6] and *Fingu*[7] drawing on a study carried out in a public pre-school in Northern Italy. The study is based on the analysis of children's interactions with these apps in the context of a sequence of activities centred on the use of the iPad. The activity fosters subitising that is the ability to quickly identify the number of items in a small set without counting.

Ladybug Count (finger mode) shows the top view of a ladybug sitting on a leaf, and the aim of each playing turn is to make the ladybug walk off the leaf. This happens when the child places on the screen (in any position) as many fingers as the dots that are on the ladybug's back. Fingu shows a room in which different kinds of floating fruits appear. The objects appear in one group or in two groups that float independently, but within each group the arrangement of the objects remains unvaried. The child has to place on the screen, simultaneously, as many fingers as the objects that are floating within a given amount of time. With these activities the ability to use fingers to represent numbers in an analogical format is fostered (Fig. 9.32).

Stellenwerttafel (place value chart) is a dynamic place value chart designed as an app for the iPad by Ulrich Kortenkamp.[8] It enables children to create tokens in a place value chart and to drag them between places. When moving a token from a place to another, the unbundling and bundling are carried out automatically. Simultaneously the token counts are displayed in the title bar (Fig. 9.33).

The behaviour of the virtual manipulative has no equivalence in the manipulation of physical MABs (see Sect. 9.3.1). MABs for units and tens are different objects (a small cube vs a column of ten small cubes).

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[6] https://itunes.apple.com/us/app/ladybug-count/id443930696?mt=8

[7] https://itunes.apple.com/en/app/fingu/id449815506?mt=8

[8] https://itunes.apple.com/de/app/stellenwerttafel/id568750442?mt=8

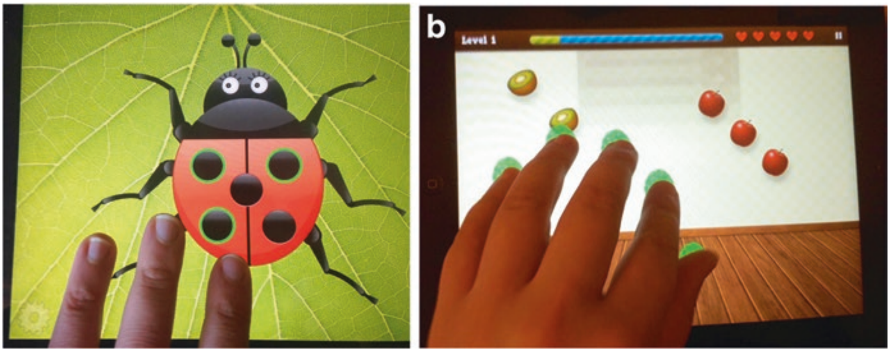
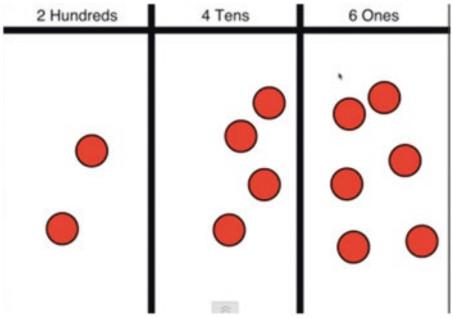


Fig. 9.32 Ladybug Count (*left*) and Fingu (*right*)

Fig. 9.33 *Stellenwerttafel* (place value chart)



Ladel and Kortenkamp (2015) have tested the use of MAB and Stellenwerttafel to build a didactical sequence for a flexible understanding of place value, in order to be able to switch between different possibilities to split a whole in parts where the parts are multiples of powers of ten. The path consists of three steps:

> In step one, the child is bundling and unbundling with base ten blocks and learns that there are ones and tens and that 10 ones have the same value as 1 ten. In step two, we introduce the place value chart with the bundling material in the title bar. The amount of ones and tens has to be illustrated by homogeneous counters (or tokens) like tally marks or points. The children learn that if the counters are homogeneous and they want to change the place they have to bundle or unbundle. In that way they connect bundling and place value by bundling in the place value chart. In step three, the children only move the counters and experience the bundling and unbundling by an automatic multiplication and division of the counters. This automation can only be provided by special virtual manipulatives. (Ladel and Kortenkamp 2015, p. 325)

This gallery of examples is by no means exhaustive. This is really a new avenue which has also the potential to be useful for low achievers.

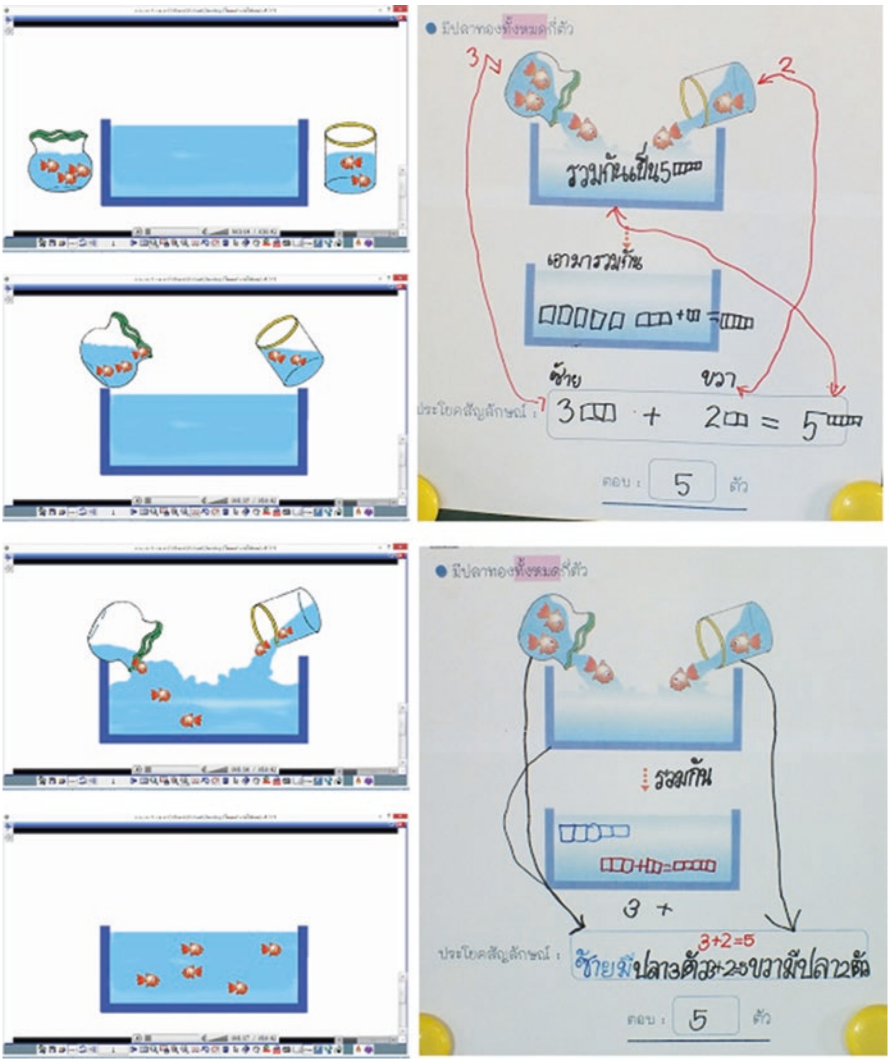


Fig. 9.34 An image from a d-book with students' drawings

9.3.5.4 Digital Textbooks

Further development in the field of digital technologies is expected in the near future. We can mention, for instance, the potential of *d-books* (*digital textbooks*[9]), developed at the University of Tsukuba, for the APEC Lesson Study Project. Digital textbooks can be created by importing existing textbooks as image files. Furthermore, interactive drawing tools can be embedded in the digital textbooks. The textbook data, together with the drawing tools, can be used interactively in classrooms. An example for primary school is Inprasitha and Jai-on (2016); see Fig. 9.34.

[9] http://math-info.criced.tsukuba.ac.jp/software/dbook/dbook_eng

9.4 Mathematical Tasks

Artefacts have the potential to foster students' construction of mathematical concepts in WNA. This potential, connected with both designers' and teachers' background and intentions, may be fulfilled by the activity. Hence, mathematical tasks come to the foreground. Some time in the working group was devoted to the discussion of task. The theme is very large, as a whole study on task design has been realised (Watson and Ohtani 2015). In this section, only a few examples are reported.

9.4.1 Cognitively Demanding Tasks

Doyle (1988 cited in Shimizu and Watanabe 2010) argues that tasks with different cognitive demands are likely to induce different kinds of learning. Mathematics tasks are important vehicles for classroom instruction that aim to enhance students' learning. To achieve quality mathematics instruction, then, the role of mathematical tasks to stimulate students' cognitive processes is crucial (Hiebert and Wearne 1993).

Kaur (2010) classified levels of cognitive demands in mathematical tasks by adapting from Stein and Smith (1998, Table 9.2).

The very high level is problem-solving, that is, the heart of mathematics. For instance, it is one of the fundamental processes in the NCTM Standards.[10] In Japan, the problem-solving approach is the preferred method for achieving the objective of teaching (Isoda 2012). Some examples of problem-solving tasks are reported in the following, using different kinds of artefacts, concrete ones (the giant Slavonic abacus) and textual ones with images and texts.

Table 9.2 Levels of cognitive demands

Levels of cognitive demand	Characteristics of tasks
Level 0 – [very low] memorisation tasks	Reproduction of facts, rules, formulas
	No explanations required
Level 1 – [low] procedural tasks without connections	Algorithmic in nature
	Focused on producing correct answers
	Typical textbook word – problems
	No explanations required
Level 2 – [high] procedural tasks with connections	Algorithmic in nature
	Has a meaningful/'real-world' context
	Explanations required
Level 3 – [very high] problem-solving/doing mathematics	Non-algorithmic in nature; requires understanding of mathematical concepts and application of
	Has a 'real-world' context/a mathematical structure
	Explanations required

[10] http://www.nctm.org/Standards-and-Positions/Principles-and-Standards/Process/

Fig. 9.35 Young children counting the beads of a giant Slavonic abacus (http://memoesperienze.comune.modena.it/bambini/index.htm)

9.4.1.1 Example: The Big Size Slavonic Abacus

We start with a short example concerning the exploration of an artefact (a giant Slavonic abacus) by pre-school students.

A big-size Slavonic abacus (Fig. 9.35; see also Sect. 9.2.2.2) with 40 beads has been introduced into more than 20 pre-schools in Modena (Italy) in the project *Bambini che contano* (Counting children, see Bartolini Bussi 2013). The way to explore it, agreed with the teachers, is based on the following questions, each of which asks the pupils to take a different perspective:

Task 1: The first impact (the narrator perspective). *What is it*? *Have you seen it before*? *What's its name*?

Task 2: The structure of the artefact (the constructor perspective). *How is it made*? *What do we need to build another one*? *How to give instructions to build another one*?

Task 3: The use of the artefact (the user perspective) to fulfil a task while playing skittles or counting the present children and similar. *How do you use it to keep the score*? *How do you use it during the call*?

Task 4: The justification for use (the mathematician perspective). *Why does it work to keep score?* and similar.

Task 5: New problems (the problem-solver perspective). As this Slavonic abacus contains only four lines (40 beads). *What to do if we needed more?*

The last task highlights that an artefact may be used even in situations that go beyond its capabilities. We might consider some hypothetical artefacts through thought experiments, which may transfer the mathematical meaning into areas that are out of reach of the original artefacts. In particular, in the example above, the children added a new line on the floor with small blocks, in order to simulate an additional line to count up to 50, if needed.

9.4.1.2 Example: A Combinatorial Task with Cuisenaire Rods

Ball and Bass (2015) have described *a demanding combinatorial task for disadvantaged students*, 'to find an order in which to list the five numbers 1,2,3,4 and 5, without repetition, in such a way that when subsets of adjacent numbers in the specific list are added together, every number, from the smallest to the largest, is possible' (p. 292). The majority of students are economically disadvantaged fifth graders. This abstract task is given to the students in a story called 'the train problem' where Cuisenaire rods (see Sect. 9.3.1.1) are used to represent cars on a 'train'. The goal of this task that is neither traditional nor standard is to encourage the experience of mathematical perseverance in solving a problem about WNA. This kind of attitude is supposed to be very important for low achievers and disadvantaged students. Other studies address motivation and learning strategies as long-term predictors of growth (Murayama et al. 2013).

9.4.1.3 Example: A Combinatorial Task with Digits

Bass suggested a *combinatorial task with digits*: Using the numbers 1, 3 and 4, each one exactly once:

- Find all the three-digit numbers you can make. How do you know that you have them all?
- Which one is largest? Smallest? How do you know?
- Which pair of them is closest together? How do you know?
- Find the sum (or the average) of all of these numbers. Can you find clever ways to do this?

9.4.1.4 Example: A Combinatorial Task on Paper and Pencil

Bass suggested another combinatorial task. In this 3 × 3 grid square, colour three of the little squares blue so that there is exactly one blue square in each row and in each column. How many ways are there to do this? How do you know that you have found them all (Fig. 9.36)?

9.4.1.5 Example: Finding Patterns on the Calendar

The calendar of October 2015 looks like the one in Fig. 9.37. The shaded part is an example of what we will call a 'square of days'. If we have any square of days, we can calculate the number *bc* – *ad*. Try a few examples. Do you notice any pattern? Do you think this is always true? If so, can you explain why? Would the same thing be true for other months?

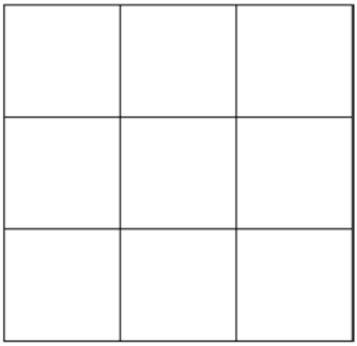


Fig. 9.36 A 3 × 3 grid square

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Fig. 9.37 The calendar

None of these examples involves a pure memorisation of procedure task. They are very high-level tasks which can be posed in pre-primary and primary school. Other examples of cognitively demanding tasks may be found in recreational mathematics (Sect. 9.2.2.6).

9.5 Artefacts and Tasks in the Institutional and Cultural Context

In the previous sections, we have considered separately artefacts (Sects. 9.2 and 9.3) and tasks (Sect. 9.4), although it is evident that they are an inseparable pair. An artefact is usually explored to contend with a mathematical task; a mathematical task is

Table 9.3 Number ranges within grades 1–4

Grade	Asia: Mainland China	Africa: Kenya	Australia	Europe: Germany	North America: USA
1	100	99	100	20	100
2	10,000	999	1000	100	1000
3	> 10,000	9999	10,000	1000	1000
4	100,000,000	99,999	>10,000	1,000,000	1,000,000

performed by some artefact (including language, text or concrete or virtual objects). In the theory of semiotic mediation (Sect. 9.2.1.2), artefacts and tasks are, together with mathematics knowledge, the element of the design process and the starting point of classroom activity. But artefacts and tasks strongly depend on the cultural and institutional constraints that determine the mathematics that is taught (Chevallard and Bosch 2014).

9.5.1 Institutional Constraints: The Number Range

The number range, in school, contributes to define the feature of tasks in WNA. It is an internationally shared approach to start with small numbers and to extend the number range in steps. In detail and application, however, there are considerable differences between countries. Examples from countries of five different continents are given in Table 9.3 (MOEST 2002; MSW 2008; CCSSO 2016; MOE 2011a, b; ACARA 2013).

While children in the other countries deal with numbers up to 100 (or 99), in Germany the focus is on understanding numbers up to 20 and on the development of addition and subtraction strategies within this number range. Although bigger numbers are used in grade 1, in some countries calculation strategies are emphasised later or are focused on numbers up to 20 or at most on special types of problems with bigger numbers (e.g. in Australia and in the USA; cf. Peter-Koop et al. 2015). By contrast, in Germany a strong tendency exists towards using the same number range both for calculation and for other activities (such as representing, composing and decomposing numbers). Tasks in textbooks and in the classrooms are usually designed according to the number range. Kenya and Australia usually use numbers (approximately) ten times bigger than in the previous grade.

Some research studies indicate considerable differences in children's whole number knowledge after completing the first grade in different countries. In a comparison of 7-year-old Australian and German children, Peter-Koop et al. (2015) refer to greater knowledge of counting and place value ideas for Australian students, whereas German children appear to develop more advanced calculating strategies. A follow-up comparison of the same children at the end of grade 2, however, indicates the same level of knowledge, especially in understanding the place value system.

9.5.2 Cultural Constraints: Language Transparency

Table 9.3 shows an incredible continuing gap between the Chinese situation and the others. It is likely that the number range for early grades is linked in a dialectical way to the most common artefacts (including abaci and language). For instance, most artefacts for place value described in the previous sections are able to represent numbers with at the utmost three (or four) digits, while the number of wires in *suàn pán* (算盘) are many more. In accordance with this, in China the number range increases faster. It is likely that there is a connection between the transparency of Chinese wording of number according to place value (this volume, Chap. 3), the faster increase of number size in primary grades and the available artefacts for representing numbers and computing.

The familiarity with big numbers in early grades does not necessarily increase the possibility of coping with more demanding mathematical tasks. According to Ni (2015):

> The strengths of Chinese children's mathematics proficiency are accompanied with notable weakness. For example, there could be an inherent problem with the curriculum system in the basic approach to mathematics thinking. Factors such as trial and error, induction, imagination and hypothesis testing are not significant part of mathematics curriculum and instruction. Probably as a consequence, for example, Chinese students appeared less tolerant for ambiguity in mathematics classroom, less willing to take risks when solving mathematical problems. The interest and confidence in learning mathematics of Chinese students was shown to deteriorate over the years as they moved up to higher grades. (p. 343)

The transparency of language may be not the only important variable. A similar issue was addressed, in a very different context, by Young-Loveridge and Bicknell (2015), who reported that, in a study about place value in New Zealand, Maori students did not perform as well as either of the other groups:

> Although the counting words used in the Maori language have a transparent decade structure, only children who are taught through the medium of Maori develop the fluency to speak and think in the Maori language. In reality, many teachers and students learn Maori as a second language, rather than being truly bilingual. (p. 383)

A further study (Theodore et al. 2015) added more results about this issue:

> We found that more Maori graduates were females than males. Previous research has also shown that Maori males are less likely to gain both tertiary and school qualifications compared to Maori females and non-Maori students, suggesting that disparities in educational participation begin early. Identified barriers to participation include lack of cultural responsiveness, difficulties transitioning from primary to secondary schooling and lower expectations of students. There were also differences found in what (e.g., commerce) and how (e.g., full-time status) Maori males studied compared to Maori females. (p. 10)

The quoted studies show that focusing only on language transparency is not enough to interpret results of research work.

9.5.3 Cultural Constraints: Bilingual Communities

Linguistic issues have been addressed by the ICMI Study 21 (Barwell et al. 2015), where specific chapters are devoted to the problems of students with the school language different from everyday language. A direct account of similar problems were discussed in the group by Veronica Sarungi (personal communication) who reported about the situation in Tanzania and other close countries (for a deeper discussion of this issue, see this volume, Chap. 3):

> The issue around language and the learning of whole numbers in Tanzania and other East African countries is complex. The diversity in the first language of learners makes teaching of mathematics in learners' first language difficult. For example, Tanzania has over 120 ethnic tribes with their own language, although these belong to major language groups such as Bantu, Nilotic and Cushitic.

Verbal language is sometimes not the best artefact to be used in classrooms. A study (Miller and Warren 2014) has analysed the performance of Australian students from disadvantaged contexts (in most cases with English as second language) showing that low performance on national numeracy testing can be overcome with a programme that focuses on specific mathematical language with rich figural representations. The importance of figural representation for Australian aboriginal children was highlighted also by Butterworth in the plenary sessions of the Conference (see also Butterworth et al. 2008; this volume Chap. 20).

9.6 Concluding Remarks: Future Challenges

In the group discussion, different designers' or teachers' intentions were identified to explain or orient the choices of the pair artefact and task. The following list is expressed in positive terms, meaning that paying attention to each issue may foster learning, while not paying attention to some issues may hinder learning. This is by no means exhaustive, but highlights some shared beliefs of the participants in the group discussion.

Epistemological Issues In this case, the mathematical consistency is in the foreground: promote students' personal reconstruction of elementary arithmetic; promote students' engagement in meaningful mathematics; promote students' flexibility between different modes of representation; promote insight.

Cognitive Issues In this case, the students' processes are in the foreground: make the maths more familiar and more user-friendly; assist mathematical inquiry, exploration, defining and proving; foster body involvement such as counting with the fingers or jumping on the number line.

Affective Issues In this case, the students' motivations and beliefs are in the foreground: create motivating learning environments; encourage, support and develop perseverance.

In this chapter we have addressed some aspects that affect whole number learning, offering examples of artefacts and mathematical tasks that may foster or hinder learning of WNA. We have collected a rich (although not complete) gallery of cultural artefacts and of teaching aids, including several realised by means of virtual technologies, and we have offered some examples of mathematical tasks concerning WNA. *Artefacts and tasks have to be considered as an inseparable pair within a given cultural and institutional context*.

We have mentioned some features of languages and cultures that sometimes hide mathematical meanings and produce the risks of didactical obstacles. The cultural roots of the epistemological obstacle represented by classical additive systems for the development of place value have been discussed.

The map is very complex. Future challenges seem to be related to teacher education. Two specific programmes for pre-primary and primary teacher education have been discussed in the working group (see Sect. 9.1.1.3), one from Canada and one from Thailand. The way of coping with epistemological and cultural issues in either programme has enriched the discussion, forcing the participants to wonder every time how teachers might cope with the complexity of designing and implementing in the classrooms teaching and learning of WNA, keeping account of the peculiar language and cultural constraints. This issue was always in the background of the discussion, although a specific panel in the Conference (this volume, Chap. 17) addresses teacher education and development.

References

Aebli, H. (1976). *Psychologische Didaktik-Didaktische Auswertungen der Psychologie von Jean Piaget*. Stuttgart: Klett-Cotta.

Australian Curriculum Assessment and Reporting Authority. (2013). *The Australian curriculum: Mathematics v2.4*. Retrieved March 17, 2013, http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10

Baccaglini-Frank, A., & Maracci, M. (2015). Multi-touch technology and preschoolers' development of number-sense. *Digital Experiences in Mathematics Education, 1*(1), 7–27.

Baccaglini-Frank, A., Hoyles, C., & Noss, R. (2014, September 21–26). From notable occurrences to situated abstractions: A window for analysing learners' thinking-in-change in a microworld. In *Proceedings of the 12th international conference of the mathematics education into the 21st century project: The future of mathematics education in a connected world.* Montenegro.

Bartolini Bussi, M. G. (2011). Artefacts and utilization schemes in mathematics teacher education: Place value in early childhood education. *Journal of Mathematics Teacher Education, 14*(2), 93–112.

Bartolini Bussi, M. G. (2013). Bambini che contano: A long-term program for preschool teachers' development. In B. Ubuz, et al. (Eds.), *CERME 8. Proceedings of the eight congress of the European Society of Research in Mathematics Education,* (pp. 2088–2097). Ankara: Middle East Technical University.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, B. Sriraman, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). New York: Routledge Taylor & Francis Group.

Bartolini Bussi, M. G., & Martignone, F. (2014). Manipulatives in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 365–372). Dordrecht: Springer.

Barwell, R., Clarkson, P., Halai, A., Kazima, M., Moschkovich, J., Planas, N., Setati, M., Valero, P., & Villavicencio Ubillús, M. (2015). *Mathematics education and language diversity*. *The 21st ICMI study* (New ICMI Studies Series). New York/Berlin: Springer.

Bruner, J. S. (1973). *The relevance of education*. New York: Norton.

Butterworth, B., Reeve, R., Reynolds, F., & Lloyd, D. (2008). Numerical thought with and without words: Evidence from indigenous Australian children. *Proceedings of the National Academy of Sciences of the United States of America, 105*(35), 13179–13184.

Cheng, Z. J. (2012). Teaching young children decomposition strategies to solve addition problems: An experimental study. *The Journal of Mathematical Behavior, 31*, 29–47.

Chevallard, Y., & Bosch, M. (2014). Didactic transposition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 170–174). Dordrecht: Springer.

Cole, M. (1996). *Cultural psychology. A once and future discipline*. Cambridge, MA: The Belknap Press.

Conti, A. (1920). *Aritmetica per la prima classe elementare,* Firenze: Bemporad e Figlio. Retrieved from http://www.indire.it/archivi/dia

D'Amore, B., Radford, L., & Bagni, G. (2016). Obstáculos epistemológicos y perspectiva sociocultural de la matemática. In B. D'Amore & L. Radford (Eds.), *Enseñanza y aprendizaje de las matemáticas: Problemas semióticos, epistemológicos y prácticos* (pp. 167–194). Bogotá: Editorial Universidad Distrital Francisco José de Caldas.

Dienes, Z. P. (1963). *An experimental study of mathematics learning*. London: Hutchinson.

Dong-Joong, K., Ferrini-Mundy, J., & Sfard, A. (2012). How does language impact the learning of mathematics? Comparison of English and Korean speaking university students' discourses on infinity. *International Journal of Educational Research, 51/52*, 86–108.

Engeström, Y. (1987). *Learning by expanding: An activity-theoretical approach to developmental research*. Helsinki: Orienta-Konsultit.

English, L. (Ed.). (1997). *Mathematical reasoning: Analogies, metaphors, and images*. Mahwah: Lawrence Erlbaum Associates.

Fauvel, J., & van Maanen, J. (Eds.). (2000). *History in mathematics education. The ICMI study*. Dordrecht: Kluwer.

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30*(2), 393–425.

Hoyles, C., & Lagrange, J. B. (Eds.). (2010). *Mathematics education and technology-rethinking the terrain. The 17th ICMI study*. New York: Springer.

Inprasitha, M., & Isoda, M. (eds). (2010). *Study with your friends: Mathematics for elementary school 1st grade*. Khon Kaen: Klungnanawittaya. (In Thai).

Inprasitha, M., & Jai-on, K. (2016). *The potential of dbook Pro to teach whole number arithmetic.* [Online]. Retrieved on March 16, 2016, from: http://www.crme.kku.ac.th/The potential of dbookPro Exemplar 3+2

Isoda, M. (2012). Problem-solving approach to develop mathematical thinking. In Isoda, M. & Katagiri, S. (Eds.), *Monographs on lesson study for teaching mathematics and sciences–Vol. 1: Mathematical thinking-how to develop it in the classroom.* (pp. 1–28). Singapore: World Scientific.

Jacobsen, L. E. (1983). Use of knotted string accounting records in old Hawaii and ancient China. *The Accounting Historians Journal, 10*(2), 53–62.

Karp, A., & Schubring, G. (Eds.). (2014). *Handbook on the history of mathematics education*. New York: Springer.

Kaur, B. (2010). A study of mathematical task from three classrooms in Singapore. In Y. Shimizu, B. Kaur, R. Huang, & D. J. Clarke (Eds.), *Mathematical tasks in classrooms around the world* (pp. 15–33). Rotterdam: Sense Publishers.

Kempinsky, H. (1921). *So rechnen wir bis hundert und darüber hinaus. Eine Anleitung für den Rechenunterricht besonders des zweiten Schuljahres*. Leipzig: Verlag der Dürr'schen Buchhandlung.

Kordemsky, B. A. (1992). *The Moscow puzzles: 359 mathematical recreations*. New York: Dover.

Lam, L. Y., & Ang, T. S. (2004). *Fleeting footsteps: Tracing the conception of arithmetic and algebra in ancient China*. Singapore: World Scientific.

Meaney, T., Trinick, T., & Fairhall, U. (2012). *Collaborating to meet language challenges in indigenous mathematics classrooms*. New York: Springer.

Menninger, K. (1969). *Number words and number symbols: A cultural history of numbers*. Cambridge, MA: The MIT Press. (Translated from the German edition of 1958).

Miller, J., & Warren, E. (2014). Exploring ESL students' understanding of mathematics in the early years: Factors that make a difference. *Mathematics Education Research Journal, 26*(4), 791–810.

Ministerium für Schule und Weiterbildung des Landes Nordrhein-Westfalen (MSW NRW). (2008). *Richtlinien und Lehrpläne für die Grundschule in Nordrhein-Westfalen. Mathematik*. Frechen: Ritterbach.

Ministry of Education Science and Technology (MOEST). (2002). *Primary education syllabus: Volume two*. Nairobi: Kenya Institute of Education.

Monaghan, J., Trouche, L., & Borwein, J. (2016). *Tools and mathematics. Instruments for learning*. New York: Springer.

Murayama, K., Pekrun, R., Lichtenfeld, S., & vom Hofe, R. (2013). Predicting long-term growth in students' mathematics achievement: The unique contributions of motivation and cognitive strategies. *Child Development, 84*(4), 1475–1490.

Ni, Y. J., Chiu, M. M., & Cheng, Z. J. (2010). Chinese children learning mathematics: From home to school. In M. H. Bond (Ed.), *The Oxford handbook of Chinese psychology* (pp. 143–154). Oxford: Oxford University Press.

Nührenbörger, M., & Steinbring, H. (2008). Manipulatives as tools in teacher education. In D. Tirosh & T. Wood (Eds.), *Tools and processes in mathematics teacher education. Volume 2 of The international handbook of mathematics teacher education* (pp. 157–182). Rotterdam: Sense Publishers.

Rabardel, R. (1995). *Les hommes et les technologies. Approche cognitive des instruments contemporains*. Paris: Armand Colin.

Radford, L. (1997). On psychology, historical epistemology and the teaching of mathematics: Towards a socio-cultural history of mathematics. *For the Learning of Mathematics, 17*(1), 26–33.

Rieber, R. W., & Wollock, J. (1997). The instrumental method in psychology. In R. W. Rieber & J. Wollock (Eds.), *The collected works of L. S. Vygotsky: Problems of the theory and history of psychology* (pp. 85–89). Boston: Springer.

Sarama, J., & Clements, D. H. (2009). "Concrete" computer manipulatives in mathematics education. *Child Development Perspectives, 3*(3), 145–150.

Schipper, W., Dröge, D., & Ebeling, A. (2000). *Handbuch für den Mathematikunterricht, 4. Schuljahr*. Hannover: Schroedel.

Shimizu, S., & Watanabe, T. (2010). Principles and processes for publishing textbooks and alignment with standards: A case in Japan. Paper presented at the *APEC conference on replicating exemplary practices in mathematics education 2010*, Koh Samul: Suratthani.

Sierpinska, A. (1996). *Understanding in mathematics*. London: Falmer Press.

Sinclair, N., & Baccaglini-Frank, A. (2015). Digital technologies in the early primary school classroom. In L. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education: Third edition* (pp. 662–686). Taylor and Francis.

Sinclair, N., & Pimm, D. (2014). Number's subtle touch: Expanding finger gnosis in the era of multi-touch technologies. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings* *of the 38th conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 209–216), Vancouver BC: PME.

Stacey, K., Helme, S., Shona Archer, S., & Caroline Condon, C. (2001). The effect of epistemic fidelity and accessibility on teaching with physical materials: A comparison of two models for teaching decimal numeration. *Educational Studies in Mathematics, 47*(2), 199–221.

Stetsenko, A. (2008). From relational ontology to transformative activist stance on development and learning: Expanding Vygotsky's (CHAT) project. *Cultural Studies of Science Education, 3*(2), 471–491.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*(4), 268–275.

Tahan, M. (1996). *O omem que calculava*. Rio de Janeiro: Grupo Editorial Record.

The Council of Chief State School Officers (CCSSO). (2016). National Governors Association Center for Best Practices and Council of Chief State School Officers. *Common core state standards for mathematics*. Retrieved March 20, 2016. http://www.corestandards.org/Math/

The Ministry of Education. (2011a). *Syllabus for the teaching of primary mathematics of the nine-year compulsory education*. 北京:北京师范大学出版社. Beijing: Beijing Normal University Press. (in Chinese).

The Ministry of Education. (2011b). *Guidelines of mathematics curriculum for 9-year compulsory education*. <基础教育课程改革纲要>. 北京:北京师范大学出版社. Beijing: People's Education Publishing. (in Chinese).

Theodore, R., Trustin, K., Kiro, C., Gollop, M., Taumoepeau, M., Taylor, N., Chee, K. S., Hunter, J., & Poulton, R. (2015, November). Maori university graduates: Indigenous participation in higher education. *Higher Education Research & Development, 2015,* 1–15. https://doi.org/10.1080/07294360.2015.1107883

Uttal, D. H., Scudder, K. V., & DeLoache, J. S. (1997). Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. *Journal of Applied Developmental Psychology, 18*(1), 37–54.

Watson, A., & Ohtani, M. (Eds.) (2015). *Task design in mathematics education*. *The 22nd ICMI study.* New York: Springer.

Wartha, S., & Schulz, A. (2012). *Rechenproblemen vorbeugen*. Berlin: Cornelsen.

Xie, X., & Carspecken, P. F. (2007). *Philosophy, learning and the mathematics curriculum dialectical materialism and pragmatism related to Chinese and American mathematics curriculums*. Rotterdam: Sense Publishers.

Zaslavsky, C. (1973). *Africa counts: Number and pattern in African cultures*. Chicago: Lawrence Hill Books.

Zhou, Z., & Peverly, S. (2005). Teaching addition and subtraction to first graders: A Chinese perspective. *Psychology in the Schools, 42*(3), 259–272.

Cited papers from Sun, X., Kaur, B., & Novotna, J. (Eds.). (2015). Conference proceedings of the ICMI study 23: Primary mathematics study on whole numbers. Retrieved February 10, 2016, from www.umac.mo/fed/ICMI23/doc/Proceedings_ICMI_STUDY_23_final.pdf

Bakker, M., van den Heuvel-Panhuizen, M., & Robitzsch, A. (2015). Learning multiplicative reasoning by playing computer games (pp. 282–289).

Baldin, Y., Mandarino, M. C., Mattos, F. R., & Guimarães, L. C. (2015). A Brazilian project for teachers of primary education: Case of whole numbers (pp. 510–517).

Ball, D. L., & Bass, H. (2015). Helping students learn to persevere with challenging mathematics (pp. 379–386).
Bartolini Bussi, M. G. (2015). The number line: A "western" teaching aid (pp. 298–306).
Cao, Y., Li, X., & Zou, H. (2015). Characteristics of multiplication teaching of whole numbers in china: The application of nine times table (pp. 423–430).
González, S., & Caraballo, J. (2015). Native American cultures tradition to whole number arithmetic (pp. 92–98).
Hodgson, B. R., & Lajoie, C. (2015). The preparation of teachers in arithmetic: A mathematical and didactical approach (pp. 307–314).
Inprasitha, M. (2015). An open approach incorporating lesson study: An innovation for teaching whole number arithmetic (pp. 315–322).
Ladel, S., & Kortenkamp, U. (2015). Development of conceptual understanding of place value (pp. 323–330).
Mercier, A., & Quilio, S. (2015). The efficiency of primary level mathematics teaching in French-speaking countries: A synthesis (pp. 331–338).
Ni, Y. J. (2015). How the Chinese methods produce arithmetic proficiency in children (pp. 341–342).
Peter-Koop, A., Kollhoff, S., Gervasoni, A., & Parish, L. (2015). Comparing the development of Australian and German children's whole number knowledge (pp. 346–353).
Pimm, D., & Sinclair, N. (2015). The ordinal and the fractional: Some remarks on a trans-linguistic study (pp. 354–361).
Roberts, N. (2015). Interpreting children's representations of whole number additive relations in the early grades (pp. 243–250).
Rottmann, T., & Peter-Koop, A. (2015). Difficulties with whole number learning and respective teaching strategies (pp. 362–370).
Siu, M. K. (2015). Pedagogical lessons from *Tongwen Suanzhi* (同文算指) – Transmission of *bisuan* (筆算 written calculation) in China (pp. 132–139).
Soury-Lavergne, S., & Maschietto, M. (2015). Number system and computation with a duo of artifacts: The pascaline and the e-pascaline (pp. 371–378).
Sun, X. (2015). Chinese core tradition to whole number arithmetic (pp. 140–148).
Young-Loveridge, J., & Bicknell, B. (2015). Using multiplication and division contexts to build place-value understanding (pp. 379–386).
Zou, D. (2015). Whole number in ancient Chinese civilisation: A survey based on the system of counting-units and the expressions (pp. 157–164).

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